An alternative subtraction scheme for NLO calculations using Nagy Soper Dipoles

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Abstract:

In this work, we present an alternative subtraction scheme for next-to-leading order QCD calculations. The subtraction terms are derived from the splitting functions of an improved parton shower with quantum interferences, and our scheme exhibits a reduced number of momentum mappings in the real emission subtraction terms and facilitates the mapping of NLO calculations with the shower. While simple cases with maximally two partons in the final state have already been presented in a previous publication, we here complete the discussion of the massless scheme by providing the final state real emission and integrated subtraction terms for processes with any number of partons in the final state. We apply our scheme to three jet production at lepton colliders at next-to-leading order and present results for the differential C-parameter distribution.

1 Introduction

With the start of data taking at the LHC in 2009 and its more than successful physics program since then, particle physics has entered an exciting era. Major tasks of the LHC experiments are the accurate measurement of the parameters of the Standard Model (SM) of particle physics, as well as the search for physics beyond the SM (BSM). For both, a precise understanding of the SM signals and background processes in an hadronic environment are crucial. These processes are mainly governed by strong interactions, where leading order (LO) calculations can exhibit uncertainties up to 100% (*c.f.* [1] for a recent review); therefore, for a correct theoretical prediction of these processes at least next-to-leading order (NLO) corrections need to be taken into account. Furthermore, it is generally not sufficient to apply an overall NLO rescaling K-factor to the leading order predictions, as NLO corrections can vary widely for different regions of phase space. More accurate predictions therefore call for the inclusion of these NLO calculations in Monte Carlo Event generators, which provide predictions for fully differential corrections to the LO process. Many such generators exist at parton level [2–18], and in recent years a lot of progress has equally been made to automatize the matching of these processes with parton showers in the Powheg [19–46] and (a)MC@NLO [22, 47–55] frameworks[1].

In this paper, we present the generic extension of an improved subtraction scheme [58–60], which facilitates the inclusion of infrared (IR) NLO divergences originating from different phase space contributions in Monte Carlo Event generators. These divergences arise whenever internal loop momenta approach zero or particles become collinear, and are known to cancel in any fixed order in perturbation theory [61, 62]. However, these cancellations occur in the combined sum of virtual and real emission contributions, and therefore originate from phase spaces with a different number of particles in the final state. In analytic or semi-numerical calculations, the singularities can be parametrized by an infinitesimal regulator; in the sum of real and virtual contributions, these regulators can then be set to zero to obtain a completely finite prediction. In numerical implementations, however, the inclusion of infinitesimal regulators can easily lead to numerical instabilities. Subtraction methods [63–73] circumvent this problem by introducing local counterterms which mimic the behaviour of the real emission matrix elements in the singular limits. The integrated counterparts of these terms are then added to the virtual contributions, where again an infinitesimal regulator is used to parametrize the singularities. Then, the higher order contributions in both phase space integrations are finite, and the regulator can be set to zero. In recent years, many of these schemes have been made available on a (semi)automated level [26, 71, 74–77].

While the behaviour of the subtraction terms in the singular limit is determined by factorization [78–80], the finite parts of the local counterterms as well as the mapping prescription between real emission and leading order phase space kinematics in the subtraction terms can differ. Unfortunately, standard schemes [65, 70] suffer from a rapidly rising number of momentum mappings, which scales like N^3 for a leading order

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[1]Recent reviews on this can be found in [56, 57].

$2 \rightarrow N$ process. Therefore, increasing the number of final state particles leads to a rapidly rising number of reevaluations of the Born matrix element. In [58–60], we therefore proposed a new subtraction scheme with a modified momentum mapping [81–83], where the number of momentum mappings scales as $\sim N^2$. The momentum mappings are constructed such that they take the whole remaining event as a spectator, and the subtraction terms are derived from the splitting functions in an improved parton shower [81–83]. In [58–60], we presented the scheme for the simplest cases with maximally two partons in the final state[2]. In the present work, we extend the scheme to cases with an arbitrary number of massless particles in the final state. We also provide the helicity dependent squared splitting functions for splittings where the mother parton is a gluon. We validate our scheme by applying it to three-jet production at NLO at lepton colliders, obtaining complete agreement with the Catani Seymour scheme.

This paper is organized as follows. In Section 2, we briefly review the generic setup for subtraction schemes. In Section 3, we review the ingredients of the new scheme and present the generalized results for the integrated subtraction terms for an arbitrary number of final state massless partons. We discuss the application of our scheme to three-jet production at lepton colliders in Section 4. Conclusion and outlook are presented in Section 5. The appendix contains a summary of the final state splitting functions [81] used as subtraction terms, a generic parametrization of four-parton phase space, and the collinear subtraction terms for processes with incoming hadrons.

2 General structure of NLO cross sections and subtraction schemes

In this section, we briefly review the general subtraction procedure for calculating NLO cross sections at lepton and hadron colliders. We start with a generic cross section at NLO

$$\sigma = \underbrace{\int_m d\sigma^B}_{\sigma^{\text{LO}}} + \underbrace{\int_{m+1} d\sigma^R + \int_m d\sigma^V}_{\sigma^{\text{NLO}}} \tag{1}$$

where σ should be specified by the respective jet function as discussed below, and $d\sigma^B$, $d\sigma^R$, and $d\sigma^V$ are the Born, real emission, and virtual contribution respectively. We here consider processes with m particles in the Born-contribution and $m+1$ partons in the real emission terms. After UV-renormalisation, the virtual and real-emission cross sections each contain infrared and collinear singularities. These cancel in the sum of virtual and real contributions [61, 62], but the individual pieces are divergent and can therefore not be integrated numerically in four dimensions.

Subtraction schemes consist of local counterterms which match the behaviour of the real-emission matrix element in the soft and collinear regions, and their integrated counterparts. Subtracting these counterterms from the real-emission matrix elements and adding back the integrated counterparts to the virtual contribution results in

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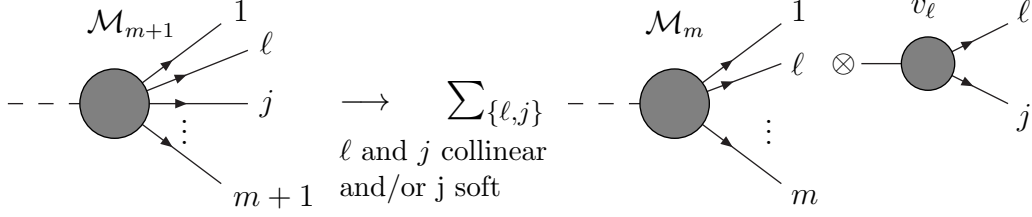[2]Some results for the generic scheme were already presented in [58].

Figure 1: Soft/collinear factorisation: when the partons ℓ and j become collinear and/or parton j becomes soft, the $(m+1)$-parton matrix element factorises into a sum over m-parton matrix elements times a singular factor v_ℓ.

finite integrands for both the virtual correction (m-particle phase space) and the real contribution ($(m+1)$-particle phase space):

$$d\sigma^{\mathrm{NLO}} \;=\; \left[d\sigma^R - d\sigma^A\right] \,+\, \left[d\sigma^A + d\sigma^V\right]. \tag{2}$$

The construction of the local counterterms, collectively denoted by $d\sigma^A$, relies on the factorisation of the real-emission matrix element in the singular (*i.e.* soft and collinear) limits (Fig. 1) [78–80], and we symbolically write

$$|\mathcal{M}_{m+1}(\hat{p})|^2 \quad \longrightarrow \quad \mathcal{D}_\ell \otimes |\mathcal{M}_m(p)|^2, \tag{3}$$

where $\mathcal{D}_\ell$ are the dipoles containing the respective singularity structure, and the symbol $\otimes$ denotes a correct convolution in color, spin, and flavour space. $\hat{p}/p$ represent momenta in $(m+1)/m$-parton phase space, respectively. As $|\mathcal{M}_{m+1}|^2$ and $|\mathcal{M}_m|^2$ live in different phase spaces, a mapping of their momenta needs to be introduced, which is defined by a mapping function F_{map} according to

$$p \;=\; F_{\mathrm{map}}\left(\hat{p}\right). \tag{4}$$

$\mathcal{D}_\ell$ and its one-parton integrated counterpart $\mathcal{V}_\ell$ are related by

$$\mathcal{V}_\ell \;=\; \int d\xi_p\, \mathcal{D}_\ell, \tag{5}$$

where $d\xi_p$ is an unresolved one parton integration measure.

In summary, any subtraction scheme needs to fulfill the following requirements:

- The dipole subtraction terms $\mathcal{D}_\ell$ must match the behaviour of the real emission matrix element in each soft and collinear region, and lead to correct IR poles when carrying out the analytical integration over the one parton phase space in a suitable regularization scheme that are necessary to cancel the soft singularities in the virtual (one-loop) matrix element,

3

- The mapping function F_{map} guarantees total energy momentum conservation as well as the on-shell condition for all external particles before and after the mapping.

Integrating Eqn. (2) over phase space using dimensional regularization [84, 85], where $D = 4 - 2\varepsilon$, then yields

$$
\begin{aligned}
\sigma^{\mathrm{NLO}} &= \underbrace{\int_{m+1} \left[d\sigma^R - d\sigma^A\right]}_{\text{finite}} + \underbrace{\int_{m+1} d\sigma^A + \int_m d\sigma^V}_{\text{finite}} \\
&= \int_{m+1} \left[d\sigma^R_{\varepsilon=0} - d\sigma^A_{\varepsilon=0}\right] + \int_m \left[\int_1 d\sigma^A + d\sigma^V\right]_{\varepsilon=0}.
\end{aligned}
\tag{6}
$$

Both integrands are now finite: the integration in $(m + 1)$ particle phase space can safely be performed in $D = 4$ dimensions, as the singular regions are regularized by the respective counterterms. In the m parton phase space, the sum of the integrated dipole contribution and the virtual correction does not contain any further poles, so that we can set $\varepsilon = 0$. Then, all integrations can be performed numerically. The explicit expressions of the cross section σ for m and $(m + 1)$ particle phase space contributions at NLO are

$$
\int_m \left[d\sigma^B + d\sigma^V + \int_1 d\sigma^A\right] = \int dPS_m \left[|\mathcal{M}_m|^2 + |\mathcal{M}_m|^2_{\text{one-loop}} + \sum_\ell \mathcal{V}_\ell \otimes |\mathcal{M}_m|^2\right],
$$

$$
\int_{m+1} \left[d\sigma^R - d\sigma^A\right] = \int dPS_{m+1} \left[|\mathcal{M}_{m+1}|^2 - \sum_\ell D_\ell \otimes |\mathcal{M}_m|^2\right],
\tag{7}
$$

where in this symbolic notation $\int dPS$ includes all flux and symmetry factors, and with $|\mathcal{M}_m|^2$, $|\mathcal{M}_{m+1}|^2$ and $|\mathcal{M}_m|^2_{\text{one-loop}}$ being the squared LO matrix element, the squared real emission matrix element and the interference term, respectively. In Eqn. (7), the sum runs over all local counterterms needed to match the complete singularity structure of the real emission contribution, and convolution with jet functions then ensures the collinear and infrared safety of the Born-level contribution. The insertion operator $I(\varepsilon)$ is then defined on a cross section level according to

$$
\int_m \int_1 d\sigma^A = \int_m d\sigma^B \otimes I(\varepsilon),
$$

where the symbol $\otimes$ again denotes a proper convolution in spin, color, and phase space. The generalization of this for processes with initial state hadrons has already been presented in [59]; for completeness, we repeat the argument in Appendix B.

Observable-dependent formulation of the subtraction method

The jet observables should be well defined such that the leading order cross sections are infrared and collinear safe. The jet cross sections are defined as

$$\sigma_J^{LO} = \int dPS_m(p_1, \cdots, p_m)\ |\mathcal{M}_m(p_1, \cdots, p_m)|^2\ F_J^{(m)}(p_1, \cdots, p_m),$$

$$\sigma_J^{NLO} = \int dPS_{m+1}(p_1, \cdots, p_{m+1})\ |\mathcal{M}_{m+1}(p_1, \cdots, p_{m+1})|^2\ F_J^{(m+1)}(p_1, \cdots, p_{m+1})$$

$$+ \int dPS_m(p_1, \cdots, p_m)\ |\mathcal{M}_m(p_1, \cdots, p_m)|^2_{\text{one-loop}}\ F_J^{(m)}(p_1, \cdots, p_m). \tag{8}$$

In general, the jet function may contain θ functions (which define cuts and corresponding cross sections) and δ functions (which define differential cross sections). For an infrared finite jet function, we require that

$$F_J^{(m+1)}(p_1, \cdots, p_j = \lambda q, \cdots, p_{m+1}) \quad \rightarrow \quad F_J^{(m)}(p_1, \cdots, p_{m+1}), \quad \text{if} \quad \lambda \to 0$$

$$F_J^{(m+1)}(p_1, .., p_i, .., p_j, .., p_{m+1}) \quad \rightarrow \quad F_J^{(m)}(p_1, .., p, .., p_{m+1})$$

$$\text{if} \quad p_i \to z\, p, \ p_j \to (1-z)\, p,$$

$$F_J^{(m)}(p_1, \cdots, p_m) \quad \rightarrow \quad 0, \quad \text{if} \quad p_i \cdot p_j \to 0. \tag{9}$$

The last condition of Eqn. (9) corresponds to an infrared safe definition of the Born-level observable, while the first two conditions guarantee infrared and collinear safety of the observables and can be summarized to

$$F_J^{(m+1)} \ \rightarrow \ F_J^{(m)} \tag{10}$$

in the singular limits.

We then have

$$\int_m \left[d\sigma^B + d\sigma^V + \int_1 d\sigma^A \right] \tag{11}$$

$$= \int dPS_m \left[|\mathcal{M}_m|^2 + |\mathcal{M}_m|^2_{\text{one-loop}} + \sum_\ell \mathcal{V}_\ell \otimes |\mathcal{M}_m|^2 \right] F_J^{(m)}(p),$$

$$\int_{m+1} \left[d\sigma^R - d\sigma^A \right] \tag{12}$$

$$= \int dPS_{m+1} \left[|\mathcal{M}_{m+1}|^2\ F_J^{(m+1)}(\hat{p}) - \sum_\ell \mathcal{D}_\ell \otimes |\mathcal{M}_m|^2\ F_J^{(m)}(p) \right],$$

$$\tag{13}$$

where in the integrated subtraction term the momenta p are derived from $\hat{p}$ using the respective momentum mapping.

3   Alternative subtraction scheme: setup

In this section, we will first review the setup of our scheme as well as the mapping and respective integration measures which have already been presented in [59, 81]. In our

scheme, the NLO subtraction terms are derived from the splitting functions introduced in a parton shower context [81–83], and the $m + 1$ to m phase space mappings needed correspond to the inverse of the respective shower m to $m + 1$ mappings. In the following, we will denote the $m + 1$ phase space four-vectors by $\hat{p}_1, \hat{p}_2, ...$ and m phase space four-vectors by $p_1, p_2, ...$. In $m+1$ phase space, the four-momenta of the emitter, emitted particle, and spectator are denoted $\hat{p}_\ell$, $\hat{p}_j$, and $\hat{p}_k$ respectively. Note that here the spectator only needs to be specified if $\hat{p}_j$ denotes a gluon, as we use the whole remaining event as a spectator in the sense of momentum redistribution for both initial and final state mappings. We here restrict our expressions to subtractions on the parton level and to massless partons.

3.1 Splitting functions

We start with a description of the matrix element factorization in the soft and collinear limits, following the notation in [81], where the QCD scattering amplitude for $m + 1$ partons is a given as a vector in (colour $\otimes$ spin) space,

$$| \, \mathcal{M}(\{\hat{p}, \hat{f}\}_{m+1})\rangle. \tag{14}$$

In the singular limits, the amplitude $| \, \mathcal{M}_\ell(\{\hat{p}, \hat{f}\}_{m+1})\rangle$ can be factorized into a splitting operator times the m-parton matrix element

$$| \, \mathcal{M}_\ell(\{\hat{p}, \hat{f}\}_{m+1})\rangle \; = \; t_\ell^\dagger(f_\ell \to \hat{f}_\ell + \hat{f}_j)\, V_\ell^\dagger(\{\hat{p}, \hat{f}\}_{m+1}) \; | \, \mathcal{M}(\{p, f\}_m)\rangle, \tag{15}$$

where the index ℓ labels the emitter/ mother parton in the $(m + 1)$/ m particle phase space. $V_\ell^\dagger(\{\hat{p}, \hat{f}\}_{m+1})$ is an operator acting on the spin part of the (colour $\otimes$ spin) space, while $t_\ell^\dagger(f_\ell \to \hat{f}_\ell + \hat{f}_j)$ is an operator acting on the colour part of the (colour $\otimes$ spin) space. The Born amplitude for producing m partons is evaluated at momenta and flavours $\{p, f\}_m$ determined from $\{\hat{p}, \hat{f}\}_{m+1}$ according to the respective momentum mappings. The spin-dependent splitting operator can be described in the spin space $| \, \{s\}_m\rangle$:

$$\langle\{\hat{s}\}_{m+1} \, | \, V_\ell^\dagger(\{\hat{p}, \hat{f}\}_{m+1}) \, | \, \{s\}_m\rangle. \tag{16}$$

If we take Eqn. (16) to be diagonal, we can define the splitting functions v_ℓ according to

$$\langle\{\hat{s}\}_{m+1} \, | \, V_\ell^\dagger(\{\hat{p}, \hat{f}\}_{m+1}) \, | \, \{s\}_m\rangle \; = \; \left(\prod_{n \notin \{\ell, j=m+1\}} \delta_{\hat{s}_n, s_n} \right) v_\ell(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_\ell, s_\ell). \tag{17}$$

Explicit forms for the splitting functions v_ℓ have been presented in [81]. For the construction of the subtraction terms, we consider the approximation for the squared matrix element in the singular limits

$$\sum_{\ell, \ell'} \langle\mathcal{M}_\ell(\{\hat{p}, \hat{f}\}_{m+1})|\mathcal{M}_{\ell'}(\{\hat{p}, \hat{f}\}_{m+1})\rangle \; \sim \; \sum_{\ell, \ell'} v_\ell^* v_{\ell'} \langle\mathcal{M}(\{p_\ell, f\}_m)|\mathcal{M}(\{p_{\ell'}, f\}_m)\rangle.$$

For the direct splitting function, where $\ell = \ell'$, we obtain

$$W_{\ell\ell} \equiv v_\ell^2 \; = \; v_\ell(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_\ell, s_\ell) \, v_\ell^*(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_\ell, s_\ell), \tag{18}$$

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which, after summing over the daughter parton spins and averaging over the mother parton spins, leads to the spin averaged splitting functions $\overline{W}_{\ell\ell}$ as subtraction terms. If the mother parton is a gluon, the Born-type matrix element might have an explicit dependence on the gluons polarization; in this case, we need to use

$$\langle\nu|W_{\ell\ell}|\nu'\rangle \tag{19}$$

in the real-emission subtraction terms, where ν, ν' are the polarisation indices of the m-parton phase space gluon. If the spin correlation tensor defined by Eqn. (19) is perpendicular to p_ℓ, the angular correlations vanish after the integration over the unresolved particles phase space and the integral over $\overline{W}_{\ell\ell}$ still provides the correct integrated counterterm [65]. For the collinear terms, the colour factors can easily be obtained [81]:

$$C(\hat{f}_\ell, \hat{f}_j) = \begin{cases} C_F & (\hat{f}_\ell, \hat{f}_j) = (q, g), (g, q), \\ C_A & (\hat{f}_\ell, \hat{f}_j) = (g, g), \\ T_R & (\hat{f}_\ell, \hat{f}_j) = (q, \bar{q}). \end{cases}$$

For soft gluon emissions, we also have to consider terms for which $\ell \neq \ell'$, which we will describe below.

3.1.1 Eikonal factor

When a gluon with four-vector $\hat{p}_j$ becomes soft, or soft and collinear with $\hat{p}_\ell$, the splitting amplitude v_ℓ defined in Eqn. (17) can be replaced by the eikonal approximation for $\hat{p}_j \rightarrow 0$

$$v_\ell^{\text{eik}}(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_\ell, s_\ell) = \sqrt{4\pi\alpha_s}\, \delta_{\hat{s}_\ell, s_\ell}\, \frac{\varepsilon(\hat{p}_j, \hat{s}_j, \hat{Q})^* \cdot \hat{p}_\ell}{\hat{p}_j \cdot \hat{p}_\ell}, \tag{20}$$

where $\varepsilon(\hat{p}_j, \hat{s}_j, \hat{Q})$ denotes the polarization vector of the emitted gluon with spin s_j. $\hat{Q}$ denotes the total momentum of the (m+1) phase space event and is used as a gauge vector. The eikonal approximation of the spin-averaged splitting functions $\overline{W}_{\ell\ell}$ is then

$$\overline{W}_{\ell\ell}^{\text{eik}} = 4\pi\alpha_s\, \frac{\hat{p}_\ell \cdot D(\hat{p}_j, \hat{Q}) \cdot \hat{p}_\ell}{(\hat{p}_j \cdot \hat{p}_\ell)^2}, \tag{21}$$

where flavour-dependent averaging factors are already taken into account. The transverse projection tensor $D^{\mu\nu}$ is given by

$$D^{\mu\nu}(\hat{p}_j, \hat{Q}) = -g^{\mu\nu} + \frac{\hat{p}_j^\mu \hat{Q}^\nu + \hat{Q}^\mu \hat{p}_j^\nu}{\hat{p}_j \cdot \hat{Q}} - \frac{\hat{Q}^2 \hat{p}_j^\mu \hat{p}_j^\nu}{(\hat{p}_j \cdot \hat{Q})^2}.$$

It will be convenient to define a dimensionless function F:

$$F = \frac{\hat{p}_\ell \cdot \hat{p}_j}{4\pi\alpha_s}\, \overline{W}_{\ell\ell}.$$

We then have

$$F_{\text{eik}} \equiv \frac{\hat{p}_\ell \cdot \hat{p}_j}{4\pi\alpha_s}\, \overline{W}_{\ell\ell}^{\text{eik}} = \frac{\hat{p}_\ell \cdot D(\hat{p}_j, \hat{Q}) \cdot \hat{p}_\ell}{\hat{p}_\ell \cdot \hat{p}_j} = \frac{2\hat{p}_\ell \cdot Q}{\hat{p}_j \cdot Q} - \frac{Q^2 \hat{p}_\ell \cdot \hat{p}_j}{(\hat{p}_j \cdot Q)^2}.$$

The eikonal factor, in combination with the interference terms, is then used to construct dipole partitioning functions.

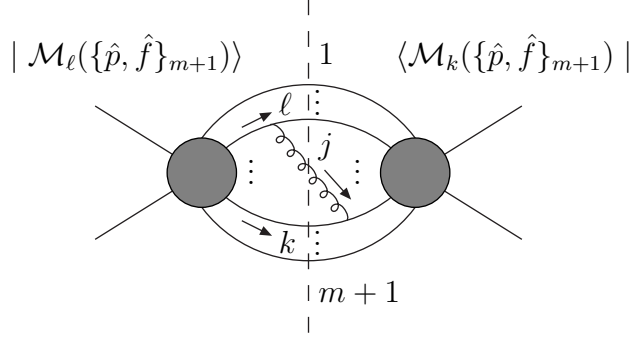Figure 2: Soft diagram: parton j is emitted from parton ℓ in the scattering amplitude and parton j is emitted from parton k in the complex-conjugate scattering amplitude.

3.1.2 Soft splitting functions

For soft gluon emission, we also need to take interference diagrams between different emitters into account. This means the emitted parton j can be emitted from emitter ℓ in the amplitude and parton j can also be emitted from a different emitter k in the complex-conjugate amplitude (Fig. 2). The interference splitting function is then given by

$$\overline{W}_{\ell k} \sim v_\ell(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_\ell, s_\ell)\, v_k(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_k, s_k)^* \, \delta_{\hat{s}_\ell, s_\ell}\, \delta_{\hat{s}_k, s_k}. \tag{22}$$

The splitting function Eqn. (22) contains a singularity when the emitted gluon j is soft; however when gluon j is collinear with parton ℓ or k, it does not contribute a leading singularity. In the special case that $\hat{p}_j$ is soft, or possibly soft and collinear with $\hat{p}_\ell$, we can use:

$$\overline{W}_{\ell k} \sim v_\ell^{\mathrm{eik}}(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_\ell, s_\ell)\, v_k^{\mathrm{eik}}(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_k, s_k)^* \, \delta_{\hat{s}_\ell, s_\ell}\, \delta_{\hat{s}_k, s_k}.$$

Note that this term only contributes if particle j is a gluon. In this prescription, there is an ambiguity in the allocation of the singularities, which can be distributed with the help of dipole partitioning functions; for completeness, we here repeat the argument in [81, 82]. The complete sum over all singular terms will contain a term

$$W_{\ell k, k\ell} = W_{\ell k}\, t_\ell^\dagger \otimes t_k \,+\, W_{k\ell}\, t_k^\dagger \otimes t_\ell.$$

For each of the two contributions, we can now introduce weight factors which redistribute the splitting functions to the corresponding mappings

$$W_{\ell k} \longrightarrow A_{\ell k} W_{\ell k}^{(\ell)} + A_{k\ell} W_{\ell k}^{(k)}, \tag{23}$$

where

$$A_{\ell k} \,+\, A_{k\ell} \,=\, 1$$

for any fixed momenta. $W^{(\ell)}$ denotes that for the mapping of this part of the interference term, $\hat{p}_\ell$ is considered to be the emitter; for $W^{(k)}$, particle $\hat{p}_k$ acts as the emitter,

such that the roles of ℓ and k are interchanged. We then have for the total sum of the two distributions

$$W_{\ell k, k \ell} = A_{\ell k} \left[W_{\ell k}^{(\ell)} t_\ell^\dagger \otimes t_k + W_{k \ell}^{(\ell)} t_k^\dagger \otimes t_\ell \right] + A_{k \ell} \left[W_{\ell k}^{(k)} t_\ell^\dagger \otimes t_k + W_{k \ell}^{(k)} t_k^\dagger \otimes t_\ell \right].$$

We now combine this with the pure squared splitting function $\overline{W}_{\ell \ell}$ with the colour factor $t_\ell^\dagger \otimes t_\ell$. Invariance of the matrix element under colour rotations implies [82]:

$$t_\ell^\dagger \otimes t_\ell = - \sum_{k \neq \ell} \frac{1}{2} \left[t_k^\dagger \otimes t_\ell + t_\ell^\dagger \otimes t_k \right],$$

and the complete contribution obeying one mapping is then given by

$$-\frac{1}{2} \left[t_k^\dagger \otimes t_\ell + t_\ell^\dagger \otimes t_k \right] \left[\overline{W}_{\ell \ell} - \overline{W}_{\ell k} \right] \tag{24}$$

with the spin averaged interference contribution

$$\overline{W}_{\ell k} = 4 \pi \alpha_s 2 A_{\ell k} \frac{\hat{p}_\ell \cdot D(\hat{p}_j, \hat{Q}) \cdot \hat{p}_k}{\hat{p}_j \cdot \hat{p}_\ell \, \hat{p}_j \cdot \hat{p}_k}. \tag{25}$$

We now split the collinear and soft parts of the respective spin-averaged splitting functions according to

$$\overline{W}_{\ell \ell} - \overline{W}_{\ell k} = \left(\overline{W}_{\ell \ell} - \overline{W}_{\ell \ell}^{\text{eik}} \right) + \left(\overline{W}_{\ell \ell}^{\text{eik}} - \overline{W}_{\ell k} \right). \tag{26}$$

The second part of Eqn. (26) can be expressed in terms of dipole partitioning functions $A'_{\ell k}$ [83]:

$$\overline{W}_{\ell \ell}^{\text{eik}} - \overline{W}_{\ell k} = 4 \pi \alpha_s A'_{\ell k} \frac{-\hat{P}_{\ell k}^2}{(\hat{p}_j \cdot \hat{p}_\ell \, \hat{p}_j \cdot \hat{p}_k)^2},$$

where $\hat{P}_{\ell k} = (\hat{p}_j \cdot \hat{p}_\ell) \, \hat{p}_k - (\hat{p}_j \cdot \hat{p}_k) \, \hat{p}_\ell$. Several choices for $A'_{\ell k}$ have been proposed in [83]; all results given here have been obtained using Eqn. (7.12) therein:

$$A'_{\ell k}(\{\hat{p}\}_{m+1}) = \frac{\hat{p}_j \cdot \hat{p}_k \, \hat{p}_\ell \cdot \hat{Q}}{\hat{p}_j \cdot \hat{p}_k \, \hat{p}_\ell \cdot \hat{Q} + \hat{p}_j \cdot \hat{p}_\ell \, \hat{p}_k \cdot \hat{Q}}.$$

The partitioning weight function $A'_{\ell k}$ also obeys the relation $A'_{\ell k}(\{\hat{p}\}_{m+1}) + A'_{k \ell}(\{\hat{p}\}_{m+1}) = 1$. The general form of the interference spin-averaged splitting function is then given by

$$\Delta W_{\ell k} = \overline{W}_{\ell \ell}^{\text{eik}} - \overline{W}_{\ell k} = 4 \pi \alpha_s \frac{2 \hat{p}_\ell \cdot \hat{p}_k \, \hat{p}_\ell \cdot \hat{Q}}{\hat{p}_\ell \cdot \hat{p}_j \left(\hat{p}_j \cdot \hat{p}_k \, \hat{p}_\ell \cdot \hat{Q} + \hat{p}_\ell \cdot \hat{p}_j \, \hat{p}_k \cdot \hat{Q} \right)} \tag{27}$$

The corresponding color factor is defined by Eqn. (24) as

$$C_{\ell k} \equiv -\frac{1}{2} \left[t_k^\dagger \otimes t_\ell + t_\ell^\dagger \otimes t_k \right]. \tag{28}$$

The only singularity in Eqn. (27) arises from the factor $\hat{p}_\ell \cdot \hat{p}_j$ in the denominator; the interference term is constructed such that it vanishes for $\hat{p}_j \cdot \hat{p}_k \to 0$. We also assume that the variables considered are such that they are finite for $p_\ell \cdot p_k \to 0$, *i.e.* singularities arising in this limit should be taken care of by the definition of the jet function as described in Section 2. The interference term only needs to be considered if the emitted parton j is a gluon. If parton j is a quark or antiquark, this term vanishes.

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3.2 Final state momentum mapping

In this section, we will describe the momentum mapping which is used in the shower prescription [81–83] as well as the subtraction scheme. As before, hatted momenta $\{\hat{p}_n\}$ are used to describe $(m+1)$-parton phase space and unhatted momenta $\{p_n\}$ m-parton phase space particles; emitter, emitted parton and spectator are labelled $\hat{p}_\ell$, $\hat{p}_j$, and $\hat{p}_k$ respectively. The four-vectors $\hat{p}_a, \hat{p}_b$ refer to initial state partons.

For a parton splitting

$$p_\ell \to \hat{p}_\ell + \hat{p}_j$$

onshellness of all momenta in both m and $(m+1)$ phase space requires a momentum mapping which reduces to

$$p_\ell = \hat{p}_\ell + \hat{p}_j$$

in the singular limits; away from these kinematic regions, an additional spectator momentum needs to be modified to guarantee $p_i^2 = \hat{p}_i^2 = 0$ for all particles. In our scheme, we use the whole remaining event as a spectator, which leads to a scaling behaviour $\sim N^2/2$ for the number of required mappings, where N is the number of final state partons in the process[3].

3.2.1 Mapping in the parton shower

For a final state splitting, we leave the momenta of the initial state partons unchanged:

$$p_a = \hat{p}_a, \quad p_b = \hat{p}_b.$$

Let Q be the total momentum of the final state partons

$$Q \equiv \sum_{n=1}^{m} p_n = p_a + p_b. \tag{29}$$

Here the momenta of the incoming partons remain the same, hence $Q = \hat{Q} = \hat{p}_a + \hat{p}_b$. We define

$$a_\ell = \frac{Q^2}{2\, p_\ell \cdot Q}, \tag{30}$$

where $a_\ell \geq 1$. The momenta of the daughter partons $\hat{p}_\ell$ and $\hat{p}_j$ are then mapped according to

$$P_\ell = \hat{p}_\ell + \hat{p}_j = \lambda\, p_\ell + \frac{1 - \lambda + y}{2\, a_\ell}\, Q. \tag{31}$$

The parameters λ and y follow from energy momentum conservation as

$$\lambda = \sqrt{(1+y)^2 - 4\, a_\ell\, y}, \qquad y = \frac{\hat{p}_\ell \cdot \hat{p}_j}{p_\ell \cdot Q}. \tag{32}$$

y is a measure for the virtuality of the splitting, with

$$y_{\max} = \left(\sqrt{a_\ell} - \sqrt{a_\ell - 1}\right)^2 = 2\, a_\ell - 1 - 2\sqrt{a_\ell\,(a_\ell - 1)} \tag{33}$$

---

[3]In the Catani Seymour scheme, each additional parton in the process subsequently serves as a spectator, leading to an overall scaling behaviour $\sim N^3/2$ for number of required mappings.

and $\lambda(y_{\mathrm{max}}) = 0$.

The mapping prescription used in our scheme now defines the whole remaining event as a spectator, *i.e.* the momenta of all corresponding final state particles are mapped as

$$\hat{p}_n^\mu = \Lambda(\hat{K}, K)^\mu{}_\nu\, p_n^\nu, \quad n \notin \{\ell, j = m + 1\} \tag{34}$$

with the Lorentz transformation

$$\Lambda^{\mu\nu}(K_1, K_2) = g^{\mu\nu} - \frac{2\,(K_1 + K_2)^\mu\,(K_1 + K_2)^\nu}{(K_1 + K_2)^2} + \frac{2\,K_1^\mu\,K_2^\nu}{K_2^2}. \tag{35}$$

K and $\hat{K}$ are given by

$$K = Q - p_\ell, \ \hat{K} = Q - P_\ell, \tag{36}$$

and correspond to the total momentum of the final state spectators before and after the splitting respectively, with

$$\hat{K}^\mu = \Lambda(\hat{K}, K)^\mu{}_\nu\, K^\nu.$$

For $a_\ell = 1$, this simplifies to

$$\hat{K} = (1 - y)\, K$$

and we therefore have

$$\Lambda(\hat{K}, K)^{\mu\nu}(a_\ell = 1) = (1 - y)\, g^{\mu\nu}. \tag{37}$$

The flavours of the spectator partons remain unchanged

$$\hat{f}_n = f_n, \quad n \notin \{\ell, j = m + 1\},$$

while the flavour of the mother parton f_ℓ obeys

$$\hat{f}_\ell + \hat{f}_j = f_\ell;$$

e.g. if the mother parton ℓ is a quark/antiquark, then we have $(\hat{f}_\ell, \hat{f}_j) = (q/\bar{q}, g)$. If the mother parton ℓ is a gluon, then $(\hat{f}_\ell, \hat{f}_j)$ can be a pair of gluons (g, g), which corresponds to $g \to g\,g$ splitting, or any choice of quark/antiquark flavours $(q, \bar{q})$, which corresponds to $g \to q\,\bar{q}$ splitting.

3.2.2 Mapping in the subtraction scheme

There is an inverse of the above mapping prescription, which maps the $(m + 1)$-parton momenta to the m-parton momenta needed for the evaluation of the real-emission subtraction terms. We start with $\{\hat{p}\}_{m+1}$ and determine $\{p\}_m$. The momentum p_ℓ of the mother parton follows directly from Eqn. (31)

$$p_\ell = \frac{1}{\lambda}\,(\hat{p}_\ell + \hat{p}_j) - \frac{1 - \lambda + y}{2\,\lambda\,a_\ell}\,Q. \tag{38}$$

The parameters y and a_ℓ read

$$y = \frac{P_\ell^2}{2\,P_\ell \cdot Q - P_\ell^2} \quad \text{and} \quad a_\ell = \frac{Q^2}{2\,P_\ell \cdot Q - P_\ell^2}, \tag{39}$$

with $P_\ell = \hat{p}_\ell + \hat{p}_j$. The parameter λ then follows from Eqn. (32).

Now we need the inverse Lorentz transformation to Eqn. (34), which is used to map all non-emitting final state spectators. We have

$$p_n^\mu = \Lambda(K, \hat{K})^\mu{}_\nu\, \hat{p}_n^\nu, \quad n \notin \{\ell, j = m+1\}, \tag{40}$$

where $\Lambda(K, \hat{K})^\mu{}_\nu$ is given by Eqn. (35). For $a_\ell = 1$, the mapping reduces to

$$p_\ell = \frac{1}{1-y}\,(P_\ell - y\,Q)\,,\; p_k = \frac{\hat{p}_k}{1-y}. \tag{41}$$

The flavour transformation is similar to the case of parton splitting. The flavour of the mother parton f_ℓ is given by

$$f_\ell = \hat{f}_\ell + \hat{f}_j,$$

with the rule of adding flavours, $q + g = q$ and $q + \bar{q} = g$. The flavours of the spectators remain unchanged:

$$f_n = \hat{f}_n, \quad n \notin \{\ell, j = m+1\}.$$

3.2.3 Phase space factorization

In the integration of the subtraction terms over the one-parton unresolved phase space, we use the generic phase-space factorisation

$$\left[d\{\hat{p}, \hat{f}\}_{m+1}\right] g(\{\hat{p}, \hat{f}\}_{m+1}) = [d\{p, f\}_m]\, d\xi_p g(\{\hat{p}, \hat{f}\}_{m+1})\,, \tag{42}$$

where $g(\{\hat{p}, \hat{f}\}_{m+1})$ is an arbitrary function. In this work, we chose to regularise the infrared and collinear singularities which appear in the splitting functions using dimensional regularisation, i.e. we work in $D = 4 - 2\varepsilon$ dimensions so that the singularities appear as $1/\varepsilon^2$ (soft and collinear) and $1/\varepsilon$ (soft or collinear) poles. We then have for the unresolved one-parton integration measure

$$d\xi_p = dy\,\theta(y_{\min} < y < y_{\max})\,\lambda^{D-3}\,\frac{p_\ell \cdot Q}{\pi}\,\frac{d^D\hat{p}_\ell}{(2\,\pi)^D}\,2\,\pi\,\delta^+(\hat{p}_\ell^2)\,\frac{d^D\hat{p}_j}{(2\,\pi)^D}\,2\,\pi\,\delta^+(\hat{p}_j^2)$$

$$\times\,(2\,\pi)^D\,\delta^{(D)}\left(\hat{p}_\ell + \hat{p}_j - \lambda\,p_\ell - \frac{1-\lambda+y}{2\,a_\ell}\,Q\right). \tag{43}$$

Here $y_{\min} = 0$ for massless partons and $y_{\max}$ is given by Eqn. (33). The reduction of this measure for the simple case $a_\ell = 1$ has been presented in [59]. In this work, we have used the parametrization[4]

$$d\xi_p = \frac{(2\,p_\ell \cdot Q)^{1-\varepsilon}}{16}\,\frac{\pi^{-\frac{5}{2}+\varepsilon}}{\Gamma\left(\frac{1}{2}-\varepsilon\right)}\times$$

$$\int_0^{y_{\max}} dy\,y^{-\varepsilon}\,\lambda^{1-2\varepsilon}\int_0^1 dz\,[z\,(1-z)]^{-\varepsilon}\int_0^1 dv\,[v\,(1-v)]^{-\frac{1+2\varepsilon}{2}}\,.$$

---

[4]We thank Z. Nagy and D. Soper for useful discussions concerning the parametrization of the integration measure.

In the center-of-mass system of $\hat{p}_\ell$, $\hat{p}_j$, where $\hat{p}_k$ defines the $x - z$ plane, z and v parametrize the polar and azimuthal angle of $\hat{p}_j$ respectively.

3.3 Generalized final state subtraction terms

In this section we present results for the subtraction terms $\mathcal{D}_\ell$ and their integrated counterparts $\mathcal{V}_\ell$ for final state emitters, where $a_\ell \neq 1$. Results for the simpler case of maximally two final state partons as well as initial state emitters have been presented in [59]. The integrated subtraction terms $\mathcal{V}_\ell$ contain integrals which depend on maximally two additional variables and need to be integrated numerically. In the expressions below, we leave out a common factor $4\pi\alpha_s$ in the expressions for the squares v_ℓ^2 of the splitting amplitudes; the (integrated) subtraction terms $\mathcal{D}_\ell$ ($\mathcal{V}_\ell$) contain all factors. We will summarize the scheme in Section 3.4. We used the Mathematica package HypExp [86, 87] in some of our calculations.

3.3.1 Parameters

In this paper, we use the labelling $\mathcal{D}_{f_\ell \hat{f}_\ell \hat{f}_j}$ and $\mathcal{V}_{f_\ell \hat{f}_\ell \hat{f}_j}$ for a process with the splitting $p_\ell \to \hat{p}_\ell + \hat{p}_j$. For final-state splittings, the subtraction terms can be expressed through the variables

$$y = \frac{\hat{p}_\ell \cdot \hat{p}_j}{p_\ell \cdot Q} \quad \text{and} \quad z = \frac{\hat{p}_j \cdot n_\ell}{P_\ell \cdot n_\ell}, \tag{44}$$

with

$$P_\ell = \hat{p}_\ell + \hat{p}_j, \; n_\ell = \frac{\gamma}{\lambda} Q - \frac{a_\ell}{\lambda} P, \; p_\ell \cdot Q = P_\ell \cdot Q - \hat{p}_\ell \cdot \hat{p}_j, \tag{45}$$

where we additionally introduced

$$\gamma = \frac{1 + \lambda + y}{2}, \; x_0 = \frac{1 - \lambda + y}{1 + \lambda + y}. $$

3.3.2 Collinear subtractions

We first consider the collinear part of the subtraction terms, which are given by the first term in Eqn. (26). These terms do not contain any soft or combined soft/ interference singularities, *i.e.* they only contain single poles $\sim \varepsilon^{-1}$ and do not depend on a specific spectator k.

qqg, q̄q̄g

The squared splitting amplitude for final state qqg couplings in the case of massless quarks is given by

$$v_{qqg}^2 = \frac{2}{y\,(p_\ell \cdot Q)} \left\{ \left[1 + \frac{(\lambda - 1 + y)^2 + 4\,y}{4\,\lambda} \right] F_{\text{eik}} + \frac{D - 2}{4} z\,[1 + y + \lambda] \right\}, \tag{46}$$

where

$$F_{\text{eik}} = 2 \left(-1 + \frac{1 + x_0}{x_0 + z\,(1 - x_0)} - \frac{x_0}{(x_0 + z\,(1 - x_0))^2} \right). \tag{47}$$

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Thus we have

$$
\begin{aligned}
\mathcal{D}_{qqg}^{\text{coll}} \;&=\; \frac{4\pi\alpha_s}{2}\, C_F\, \left(v_{qqg}^2 - v_{\text{eik}}^2 \right) \\
&=\; \frac{4\,\pi\,\alpha_s}{y\,(p_\ell\cdot Q)}\, C_F\, \left\{ \frac{(\lambda - 1 + y)^2 + 4\,y}{4\,\lambda}\, F_{\text{eik}} \;+\; \frac{D-2}{4}\, z\,[1 + y + \lambda] \right\}, \quad (48)
\end{aligned}
$$

and the integrated subtraction term is

$$
\begin{aligned}
\mathcal{V}_{qqg}^{\text{coll}} \;=\; &\frac{\alpha_s}{4\,\pi}\, C_F\, \frac{1}{\Gamma(1-\varepsilon)}\, \left(\frac{2\,\pi\,\mu^2}{p_\ell\cdot Q} \right)^{\varepsilon} \times \left\{ -\frac{1}{\varepsilon} + 4\,I_3(a_\ell) \right. \\
&+\frac{1}{2}\, [(9 - 7a_\ell)(a_\ell - 1)\log(a_\ell - 1) + a_\ell(7a_\ell - 16)\log(a_\ell) - 7\log(y_{\max}) \\
&\left. -a_\ell(2y_{\max} + 7) - 7y_{\max} - 4] \right\}, \quad (49)
\end{aligned}
$$

with

$$
I_3(a_\ell) \;=\; -\int_0^{y_{\max}} dy\, \left[\frac{(\lambda - 1 + y)^2}{4\,y} + 1 \right] \frac{(1+y)\ln x_0}{\lambda}. \quad (50)
$$

gq$\bar{q}$, g$\bar{q}$q

The $gq\bar{q}$ splitting function for massless quarks, keeping the gluon helicity for the mother parton, is given by

$$
\langle \nu | v_{gq\bar{q}}^2 | \nu' \rangle \;=\; \frac{1}{\hat{p}_\ell\,\hat{p}_j}\, \left[-g_{\nu\,\nu'} - 2\frac{k_{\perp,\nu}\,k_{\perp,\nu'}}{\hat{p}_\ell\,\hat{p}_j} \right],
$$

where $k_\perp$ can easily be obtained from a Sudakov parametrization as

$$
k_\perp \;=\; \hat{p}_\ell - \frac{1}{\lambda}\, \left[P_\ell\,\gamma\,(1 - z\,(1 + x_0)) + y\,Q\,(2\,z - 1) \right], \quad (51)
$$

with $k_\perp \cdot p_\ell = k_\perp \cdot n_\ell = 0$. If there is no explicit helicity dependence in the Born-type matrix element, we have

$$
v_{gq\bar{q}}^2 \;=\; \frac{2}{y\,p_\ell Q}\,(1 - \varepsilon - 2\,z\,(1-z)). \quad (52)
$$

We obtain for the subtraction terms

$$
\langle \nu | \mathcal{D}_{gqq} | \nu' \rangle \;=\; 4\pi\alpha_s\, T_R \langle \nu | v_{gq\bar{q}}^2 | \nu' \rangle, \quad \mathcal{D}_{gqq}^{\text{av}} \;=\; \frac{4\pi\alpha_s}{2\,(1-\varepsilon)}\, T_R\, v_{gq\bar{q}}^2. \quad (53)
$$

Integrating this over the unresolved one-parton phase-space yields

$$
\mathcal{V}_{gqq}(a_\ell) \;=\; \frac{T_R}{\pi}\, \frac{\alpha_s}{\Gamma(1-\varepsilon)}\, \left(\frac{2\,\pi\,\mu^2}{p_\ell\cdot Q} \right)^{\varepsilon} \left[-\frac{1}{3\,\varepsilon} - \frac{8}{9} + \frac{1}{3}\,[(a_\ell - 1)\,\ln(a_\ell - 1) - a_\ell\,\ln a_\ell] \right]. \quad (54)
$$

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ggg

The total (unaveraged) splitting amplitude squared, in the helicity basis of the mother parton p_ℓ, is given by

$$\langle \nu | v_{ggg}^2 | \nu' \rangle \;=\; \frac{1}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2} \left\{ -g^{\nu\nu'} (\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell \,+\, \hat{p}_j \cdot D_\ell \cdot \hat{p}_j) + k_\perp^\nu \, k_\perp^{\nu'} \, \mathrm{Tr}[D_\ell \cdot D_j] \right\},$$
(55)

with

$$\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell \;=\; \frac{2\,y\,p_\ell \cdot Q}{x_0 + z\,(1 - x_0)} \left[1 \,-\, z\,(1 - x_0) \,-\, \frac{x_0}{x_0 + z\,(1 - x_0)} \right],$$

$$\hat{p}_j \cdot D_\ell \cdot \hat{p}_j \;=\; \frac{2\,y\,p_\ell \cdot Q}{1 - z\,(1 - x_0)} \left[x_0 \,+\, z\,(1 - x_0) \,-\, \frac{x_0}{1 - z\,(1 - x_0)} \right],$$

$$\mathrm{Tr}\,[D_\ell \cdot D_j] \;=\; D - 2 - 2\,\Delta + \Delta^2$$

and

$$\Delta \;=\; \frac{\hat{Q}^2\,(\hat{p}_\ell \cdot \hat{p}_j)}{(\hat{p}_\ell \cdot \hat{Q})\,(\hat{p}_j \cdot \hat{Q})} \;=\; \frac{2\,x_0}{(x_0 + z\,(1 - x_0))\,(1 - z\,(1 - x_0))}.$$
(56)

and $k_\perp$ again given by Eqn. (51); if the Born matrix element is helicity independent, we have

$$v_{ggg}^2 \;=\; \frac{1}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2} \left\{ (D - 2)\,[\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell + \hat{p}_j \cdot D_\ell \cdot \hat{p}_j] \,-\, k_\perp^2 \,\mathrm{Tr}\,[D_\ell \cdot D_j] \right\},$$

with

$$k_\perp^2 \;=\; -2\,y\,z\,(1 - z)\,p_\ell \cdot Q.$$
(57)

Instead of using this as a subtraction term, however, we proceed in a different way, and define a subtraction term that only contains soft singularities from particle j [82]: We introduce

$$\langle \nu | v_{ggg,\mathrm{sub}}^2 | \nu' \rangle \;=\; \langle \nu | v_2^2 - v_3^2 | \nu' \rangle \;=\; -\frac{g^{\nu\nu'}}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2} \,[\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell - \hat{p}_j \cdot D_\ell \cdot \hat{p}_j],$$
(58)

where $v_{2,3}$ are defined corresponding to Eqns. (2.40)-(2.42) in [82]. This leads to

$$\langle \nu | \tilde{v}_{ggg}^2 | \nu' \rangle \;=\; \langle \nu | v_{ggg}^2 + v_{ggg,\mathrm{sub}}^2 | \nu' \rangle$$

$$=\; \frac{1}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2} \left\{ -2\,g^{\nu\nu'} \,\hat{p}_\ell \cdot D_j \cdot \hat{p}_\ell \,+\, k_\perp^\nu \, k_\perp^{\nu'} \mathrm{Tr}\,[D_\ell \cdot D_j] \right\},$$

which is the subtraction term for each gluon emission. The first part is the unaveraged eikonal splitting function; if we combine this with the interference term, we have

$$\langle \nu | \tilde{v}_{ggg}^2 - v_{\mathrm{eik}}^2 | \nu' \rangle \;=\; \frac{k_\perp^\nu \, k_\perp^{\nu'}}{2\,(\hat{p}_\ell \cdot \hat{p}_j)^2} \,[D - 2 - \Delta(2 - \Delta)].$$
(59)

The collinear subtraction term reads

$$\langle \nu | D_{ggg}^{\mathrm{coll}} | \nu' \rangle \;=\; \frac{2\,\pi\,\alpha_s}{1 - \varepsilon} \, C_A \, \frac{k_\perp^\nu \, k_\perp^{\nu'}}{(\hat{p}_\ell \cdot \hat{p}_j)^2} \,[D - 2 - \Delta(2 - \Delta)].$$
(60)

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If there is no angular correlation in the Born-type matrix element, we can replace $k_\perp^\nu \, k_\perp^{\nu'} \to -k_\perp^2$ in the above expressions, and equally need to multiply by $1/2(1-\varepsilon)$.

Note that the above reshuffling of singular terms requires that for a final state with $g(\hat{p}_1)g(\hat{p}_2)$, both combinations $(i,j) = (1,2),(2,1)$ need to be taken into account; the factor $\frac{1}{2}$ which is included in Eqn. (55) and all subsequent expressions guarantees a correct mapping of the singularity structure.

Integrating and taking all averaging factors into account gives

$$
\begin{aligned}
\mathcal{V}_{ggg}^{\text{coll}} &= \mu^{2\varepsilon} \frac{4\pi\alpha_s}{2\,(1-\varepsilon)} C_A \int d\xi_p \, \left(\tilde{v}_{ggg}^2 - v_{\text{eik}}^2\right) \\
&= \left(\frac{2\pi\mu^2}{p_\ell \cdot \hat{Q}}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \frac{\alpha_s}{2\pi} C_A \left[-\frac{1}{6\varepsilon} - \frac{4}{9} + \frac{1}{6}\left[(a_\ell-1)\ln(a_\ell-1) - a_\ell\ln a_\ell\right] + I_{\text{fin}}(a_\ell)\right]
\end{aligned}
\tag{61}
$$

with

$$
\begin{aligned}
I_{\text{fin}}(a_\ell) \;\overset{a_\ell \neq 1}{=}\; & a_\ell \left\{ 1 - \sqrt{a_\ell}\ln\left(\frac{\sqrt{a_\ell}+1}{\sqrt{a_\ell}-1}\right) - \ln\left(\frac{a_\ell}{a_\ell-1}\right)\right. \\
& \left. + 8\,a_\ell \int_0^{y_{\max}} dy \, \frac{y\ln x_0}{\lambda^2\,(1+y)^3}\left[a_\ell\,y - (1+y)^2\right]\right\} \\
\overset{a_\ell=1}{=}\; & -\frac{3}{8}\pi^2 + \frac{7}{2}.
\end{aligned}
\tag{62}
$$

3.3.3   Soft and soft/collinear subtractions

We now discuss the integration of the interference term, which is given by the second contribution in Eqn. (26). This term does not depend on the specific nature of the splitting, *i.e.* it is universal; it contains all soft and soft/ colliner singularities and equally depends on a spectator parton k. Parton j needs to be a gluon, otherwise this contribution vanishes.

We start from the definition of the interference term in Eqn. (27):

$$
\frac{1}{4\pi\alpha_s}\Delta W_{\ell k} = \frac{2\,(\hat{p}_\ell \cdot \hat{p}_k)\,(\hat{p}_\ell \cdot \hat{Q})}{(\hat{p}_\ell \cdot \hat{p}_j)\left((\hat{p}_j \cdot \hat{p}_k)\,(\hat{p}_\ell \cdot \hat{Q}) + (\hat{p}_\ell \cdot \hat{p}_j)(\hat{p}_k \cdot \hat{Q})\right)}
$$

and the subtraction term

$$
\mathcal{D}^{\text{if}}(\hat{p}_\ell, \hat{p}_j, \hat{p}_k) = C_{\ell k}\,\Delta W_{\ell k},
\tag{63}
$$

where $C_{\ell k}$ is given by Eqn. (28). Note that the above expression holds also for cases where the mother parton is a gluon, as the interference terms are diagonal in helicity space[5].

---

[5]*I.e.* for helicity dependent Born-type matrix elements $\mathcal{M}$, where a spin correlation tensor $\mathcal{T}^{\mu\nu}$ is

We obtain for the integrated subtraction term

$$
\begin{aligned}
\mathcal{V}_{\ell k}^{\mathrm{if}} \;=\; & \mu^{2\varepsilon} C_{\ell k} \int d\xi_p \left(\Delta W_{\ell k}\right) = \left(\frac{2\,\mu^2\,\pi}{p_\ell \cdot Q}\right)^{\varepsilon} \frac{\alpha_s}{\pi} \frac{1}{\Gamma(1-\varepsilon)} C_{\ell k} \\
& \times \left\{ \frac{1}{2\,\varepsilon^2} + \frac{1}{\varepsilon}\left[1 + \frac{1}{2}\ln\left(\tilde{a}_0^{(\ell k)} + a_\ell\right)\right] - \frac{\pi^2}{6} + 3 - 2\ln 2 \ln\left(\tilde{a}_0^{(\ell k)} + a_\ell\right) \right. \\
& + \frac{1}{\pi}\left[I_{\mathrm{fin}}^{(b)}\left(\frac{\tilde{a}_0^{(\ell k)}}{a_\ell}\right) + I_{\mathrm{fin}}^{(d)}(a_\ell, \tilde{a}^{(\ell k)}) + I_{\mathrm{fin}}^{(e)}(a_\ell) \right] \\
& \left. + \ln a_\ell \left[2\ln 2 - \frac{1}{4}\ln a_\ell + \frac{1}{2}\ln\left(\tilde{a}_0^{(\ell k)} + a_\ell\right) + 1\right] \right\},
\end{aligned}
$$

$$(64)$$

with

$$
\begin{aligned}
I_{\mathrm{fin}}^{(b)}(b) \;=\; & \frac{\pi}{2}\left[\int_0^1 \frac{du}{u}\left\{ 2\ln 2 + \frac{1}{\sqrt{1 + 4\,b(1+b)\,u^2}} \times \ln\left[\frac{(1-u)}{\left(1 + 2\,b\,u + \sqrt{1 + 4\,b\,(1+b)\,u^2}\right)^2} \right] \right\} \right. \\
& \left. + 2\ln 2 \ln(1+b) + \frac{1}{2}\ln^2(1+b) + \frac{5}{2}\mathrm{Li}_2\left(\frac{b}{b+1}\right) - \frac{1}{2}\mathrm{Li}_2\left[\left(\frac{b}{b+1}\right)^2\right] \right],
\end{aligned}
$$

$$
\begin{aligned}
I_{\mathrm{fin}}^{(d)}(a_\ell, \tilde{a}^{(\ell,k)}) \;=\; & \pi \int_0^1 \frac{du}{u} \int_0^1 \frac{dx}{x} \\
& \times \left\{ \gamma_\ell\, x \left[\frac{1 - x + x_{0,\ell}\left[\lambda_\ell \frac{\tilde{a}^{(\ell,k)}}{a_\ell} + 2\right]}{\sqrt{\left[A^{(\ell k)}\left(1 + x_{0,\ell} - x\right) + x_{0,\ell}\left(\lambda_\ell \frac{\tilde{a}^{(\ell,k)}}{a_\ell} + 1\right)\right]^2 - [B^{(\ell k)}]^2(1 + x_{0,\ell} - x)^2}} - 1 \right] \right. \\
& \left. + x - \frac{1}{\sqrt{1 + 4\,\frac{u^2\,\tilde{a}_0^{(\ell,k)}}{a_\ell}\left[1 + \frac{\tilde{a}_0^{(\ell,k)}}{a_\ell}\right]}} \right\}
\end{aligned}
$$

$$
I_{\mathrm{fin}}^{(e)}(a_\ell) \;=\; \pi \int_0^1 dx\, \frac{1-x}{x} \ln\left[\frac{\delta_\ell\, a_\ell}{x}\right].
$$

$$(65)$$

We have introduced

$$
A^{(\ell k)}(p_\ell, p_k) \;=\; z_\ell\, \gamma_\ell\, \frac{p_\ell\, \hat{p}_k}{\hat{p}_k\, P_\ell} + (1 - z_\ell)\, \tilde{z}^{(\ell k)}, \quad B^{(\ell k)}(p_\ell, p_k) \;=\; 2\,\sqrt{z_\ell\, \tilde{z}^{(\ell k)}\,(1 - z_\ell)\,(1 - \tilde{z}^{(\ell k)})},
$$

defined such that $-g_{\mu\nu}\mathcal{T}^{\mu\nu} = |\mathcal{M}|^2$, the interference subtraction term is given by

$$
-g_{\mu\nu} D^{\mathrm{if}} \mathcal{T}^{\mu\nu} \;=\; D^{\mathrm{if}} |\mathcal{M}|^2.
$$

and

$$z_\ell = \frac{x - x_{0,\ell}}{1 - x_{0,\ell}}, \qquad \tilde{z}^{(\ell k)} = \frac{y_\ell}{\gamma_\ell} \tilde{a}^{(\ell k)},$$

$$\tilde{a}^{(\ell k)} = \frac{\hat{p}_k \cdot n_\ell}{\hat{p}_k \cdot P_\ell}, \qquad \tilde{a}_0^{(\ell k)} = \tilde{a}^{(\ell k)}(y_\ell = 0) = \frac{p_k \cdot n_\ell}{p_k \cdot P_\ell}$$

$$\delta_\ell(x) = \frac{a_\ell - \sqrt{a_\ell^2 - a_\ell \frac{4x}{(1+x)^2}}}{2x}(1+x)^2 - 1,$$

where y_ℓ in all above expressions is defined by[6]

$$y_\ell := \delta_\ell(x)\, u.$$

We here made the dependence on the momenta p_ℓ, p_k explicit in the labelling of the variables λ, γ, x_0, ..., which are all defined according to Sections 3.2 and 3.3.1 respectively.

In terms of the Born-type kinematics, P_ℓ can easily be recovered from Eqn. (31); $\hat{p}_k$ needs to be reconstructed in the Born-type integrations according to

$$\hat{p}_k = \Lambda(\widehat{K}, K)\, p_k, \qquad (66)$$

with the Lorentz transformation defined according to Eqn. (35) and with

$$K = Q - p_\ell, \quad \widehat{K} = Q\left(1 - \frac{\gamma_\ell x_{0,\ell}}{a_\ell}\right) - \lambda_\ell\, p_\ell.$$

Finally, note that

$$a_\ell + \tilde{a}_0^{(\ell,k)} = \frac{p_k \cdot Q}{p_k \cdot p_\ell}.$$

3.4   Final expressions

In this section, we describe how the expressions in the last subsections should be combined to provide the subtraction terms $d\sigma_A$ and their integrated counterparts $\int_1 d\sigma_A$.

The complete parton level contribution is given by the sum of $\sigma_{ab}^{\mathrm{LO}}$ and $\sigma_{ab}^{\mathrm{NLO}}$, with

$$\sigma_{ab}^{\mathrm{LO}} = \int_m d\sigma_{ab}^B(p_a, p_b),$$

$$\sigma_{ab}^{\mathrm{NLO}} = \int_{m+1} d\sigma_{ab}^R(\hat{p}_a, \hat{p}_b) + \int_m d\sigma_{ab}^V(p_a, p_b) + \int_m d\sigma_{ab}^C(p_a, p_b, \mu_F^2).$$

The NLO contribution can be split into

$$\sigma_{ab}^{\mathrm{NLO}} = \int_{m+1} \left[d\sigma_{ab}^R(\hat{p}_a, \hat{p}_b) - d\sigma_{ab}^A(\hat{p}_a, \hat{p}_b) \right]$$

$$+ \int_m \left[\int d\sigma_{ab}^V(p_a, p_b) + \int_1 d\sigma_{ab}^A(\hat{p}_a, \hat{p}_b) + d\sigma_{ab}^C(p_a, p_b, \mu_F^2) \right]_{\varepsilon=0},$$

---

[6]We thank Z. Nagy for providing us with this variable transformation for the interference terms.

where $\int_1 d\sigma_{ab}^A + d\sigma_{ab}^C$ can be written as

$$\int_m \left[\left[\int_1 d\sigma_{ab}^A(\hat{p}_a, \hat{p}_b) + d\sigma_{ab}^C(p_a, p_b, \mu_F^2) \right] \right.$$

$$= \int_m d\sigma_{ab}^B(p_a, p_b) \otimes I(\varepsilon) + \int_0^1 dx \int_m d\sigma_{ab}^B(x\hat{p}_a, p_b) \otimes \left[K^a(x\,\hat{p}_a) + P(x, \mu_F^2) \right]$$

$$+ \int_0^1 dx \int_m d\sigma_{ab}^B(\hat{p}_a, x\hat{p}_b) \otimes \left[K^b(x\hat{p}_b) + P(x, \mu_F^2) \right],$$

where the insertion terms K, P only appear in the case of initial state partons, *c.f.* Appendix B. All observables, as well as infrared safety of the Born level contributions, need to be introduced in terms of jet functions as discussed in Section 2, *c.f.* Eqn. (11). For an incoming lepton, the collinear counterterm is set to zero and the PDF is replaced by a structure function $f_{i/I}^{\text{ew}} = \delta(1 - \eta_i)$.

In the following, we discuss the specific form of $d\sigma_{ab}^A(p_a, p_b)$ which corresponds to the subtraction term in the real emission contribution of the process, as well as the integrated D-dimensional counterterm $\int_1 d\sigma_{ab}^A(p_a, p_b)$. In general, the subtraction term can be split into contributions originating from all possible emitters $\hat{p}_\ell$ [7]:

$$d\sigma_{ab}^A(\hat{p}_a, \hat{p}_b) = \sum_\ell d\sigma_{ab}^{A,\ell}(\hat{p}_a, \hat{p}_b), \tag{67}$$

where $\hat{p}_\ell$ can denote an initial or final state particle. We have for each contribution

$$d\sigma_{ab}^{A,\ell}(\hat{p}_a, \hat{p}_b) = \frac{N_{m+1}}{\Phi_{m+1}} \int_{m+1} \sum_{j \neq \ell} \mathcal{D}_{f_\ell \hat{f}_\ell \hat{f}_j}(\hat{p}_\ell, \hat{p}_j) \otimes |\mathcal{M}(p)|_{m;f_\ell}^2, \tag{68}$$

where $|\mathcal{M}|_{m;f_\ell}^2$ denotes the squared Born matrix element with a flavour f_ℓ of the mother parton; the extension for cases where there is an angular dependence of the Born-type matrix element is straightforward. The momenta $\{p_m\}$ are determined from $\{\hat{p}_m\}$ through the respective mapping. N_{m+1} incorporates all symmetry factors of the $m+1$ process and $\Phi_{m+1} = 2\,\hat{s}$ is the respective flux factor. For splittings where the mother parton is a gluon, we use the following conventions: for $g \to q\,\bar{q}$ final state splittings, we always choose $(\hat{f}_\ell, \hat{f}_j) = (q, \bar{q})$; for $g \to gg$, *i.e.* a final state which contains $g(\hat{p}_1)g(\hat{p}_2)$, we need to consider *both* combinations $(\hat{p}_\ell, \hat{p}_j) = (\hat{p}_1, \hat{p}_2), (\hat{p}_2, \hat{p}_1)$; we compensate this by introducing an additional factor $\frac{1}{2}$ in the respective (integrated) subtraction terms. This factor has already been accounted for in all expressions in Section 3.3.

The subtraction terms can be split into collinear and interference terms:

$$\mathcal{D}_{f_\ell \hat{f}_\ell \hat{f}_j}(\hat{p}_\ell, \hat{p}_j) = \mathcal{D}_{f_\ell \hat{f}_\ell \hat{f}_j}^{\text{coll}}(\hat{p}_\ell, \hat{p}_j) + \delta_{\hat{f}_j, g} \sum_{k \neq (\ell, j)} \mathcal{D}^{\text{if}}(\hat{p}_\ell, \hat{p}_j, \hat{p}_k), \tag{69}$$

where $\mathcal{D}^{\text{if}}(\hat{p}_\ell, \hat{p}_j, \hat{p}_k)$ now denotes an interference contribution where $\hat{p}_k$ acts as a spectator as discussed in Section 3.1.2. Note that there is a *unique momentum mapping*

---

[7]In the following, we omit the jet functions for notational reasons; however, full expressions should always be read according to Eqn. (11) where all jet functions are included.

for each combination $(\hat{p}_\ell, \hat{p}_j)$ which is the same for all interference terms appearing in $\mathcal{D}_{f_\ell \hat{f}_\ell \hat{f}_j}(\hat{p}_\ell, \hat{p}_j)$.

The integrated counterterms are given by the integrated form of Eqn. (67):

$$\int_1 d\sigma_{ab}^A(\hat{p}_a, \hat{p}_b) = \int_1 \sum_\ell d\sigma_{ab}^{A,\ell}(\hat{p}_a, \hat{p}_b).$$

The collection of the integrated counterterms is then straightforward: for each dipole which has been subtracted in the real emission part, the respective integrated contribution to I, K, P needs to be added to the virtual contribution as in Eqn. (100). Finally, our expressions have been derived on a matrix element level:

$$\int_1 |\mathcal{M}|_{m+1}^2 \to \int_1 \mathcal{D} \otimes |\mathcal{M}|_m^2 = \mathcal{V} \otimes |\mathcal{M}|_m^2;$$

on cross section level, we additionally have to take the flux as well as combinatoric factors into account[8]

$$\int_1 d\sigma_{m+1;ab}^A(\hat{p}_a, \hat{p}_b) = \frac{N_{m+1}}{2\hat{s}} \int_1 \mathcal{D} \otimes |\mathcal{M}|_m^2 = \frac{N_{m+1}}{2\hat{s}} \mathcal{V} \otimes |\mathcal{M}|_m^2,$$

$$\int_m \int_1 d\sigma_{m+1;ab}^A = N_{m+1} \int_m \frac{1}{2\hat{s}} \mathcal{V} \otimes |\mathcal{M}|_m^2 = \frac{N_{m+1}}{N_m} x_s \mathcal{V} \otimes \int_m d\sigma_m,$$

where the factors N_m, N_{m+1} account for possible symmetry factors of the specific process, and where here $x_s = s/\hat{s}$ is the ration of the partonic center-of-mass energies before and after the splitting; $x_s = 1$ for final state emitters. We then obtain the relation

$$\sum \mathcal{V} = \frac{1}{x_s} \frac{N_m}{N_{m+1}} (I + K + P) \tag{70}$$

between the integrated splitting functions $\mathcal{V}$ given in the next sections and the insertion operators I, K, P.

4 Example: $e^+ e^- \to$ 3 jets

In this section we consider the simplest non-trivial process with more than two partons in the final state: three-jet production in $e^+ e^-$ annihilation. The next-to-leading order contributions to this process are well-known [63, 88–90]. We compare the results obtained from the implementation of our scheme and from a private implementation of the Catani Seymour scheme as well as [90][9]. We find complete agreement for the differential C-parameter [63], with integration errors on the percent level.

The leading order process we consider is given by

$$e^+ e^- \longrightarrow q(p_1)\, \bar{q}(p_2)\, g(p_3). \tag{71}$$

---

[8]Correct counting of symmetry factors needs to be done explicitly in this expression; if all splitting multiplicities and symmetry factors are taken into account, we obtain a generic combinatoric factor $\frac{1}{2}$ for ggg splittings, c.f. Section 7.2 in [65].

[9]We thank M. Seymour for help with the original code available from [91].

At next-to-leading order, two different real-radiation subprocesses contribute:

$$
\begin{aligned}
(A) && e^+ e^- &\to \gamma^*(Q) \to q(\hat{p}_1)\, \bar{q}(\hat{p}_2)\, g(\hat{p}_3)\, g(\hat{p}_4), \\
(B) && e^+ e^- &\to \gamma^*(Q) \to q(\hat{p}_1)\, \bar{q}(\hat{p}_2)\, q(\hat{p}_3)\, \bar{q}(\hat{p}_4).
\end{aligned}
\tag{72}
$$

For a complete next-to-leading order calculation, the virtual corrections need to be added to the leading order contribution. In the following, we use the notation

$$
x_i = 2\, p_i \cdot Q/Q^2,\ y_{ij} = s_{ij}/Q^2,\ s_{ijk} = s_{ij} + s_{ik} + s_{jk},
$$

with $s_{ij} = (p_i + p_j)^2$. Energy-momentum conservation leads to

$$
\sum_{i,j>i} y_{ij} = 1,\ \sum_i x_i = 2.
$$

We equally follow the notation for matrix elements in Section 3.1:

$$
\begin{aligned}
\langle \{p_i\} \mid \{p_i\} \rangle &= \langle \mathcal{M}_3\left(\{p_i\}\right) \mid \mathcal{M}_3\left(\{p_i\}\right) \rangle \equiv \left| \mathcal{M}_3\left(\{p_i\}\right) \right|^2, \\
\langle \{\hat{p}_i\} \mid \{\hat{p}_i\} \rangle &= \langle \mathcal{M}_4\left(\{\hat{p}_i\}\right) \mid \mathcal{M}_4\left(\{\hat{p}_i\}\right) \rangle \equiv \left| \mathcal{M}_4\left(\{\hat{p}_i\}\right) \right|^2
\end{aligned}
$$

The total next-to-leading order contribution for process (71) is then given by:

$$
\begin{aligned}
\sigma_J^{NLO} = &\int dPS_4 \left[\left| \mathcal{M}_4\left(\{\hat{p}_i\}\right) \right|^2 F_J^{(4)}\left(\{\hat{p}_i\}\right) - \sum_{\ell,j} \langle q,r,s | \mathcal{D}_{\ell j}\left(\{\hat{p}_i\}\right) | q,r,s \rangle F_J^{(3)}(q,r,s) \right]_A \\
&+ \int dPS_4 \sum_{\text{flavours}} \left[\left| \mathcal{M}_4\left(\{\hat{p}_i\}\right) \right|^2 F_J^{(4)}\left(\{\hat{p}_i\}\right) - \sum_{\ell,j} \langle q,r,s | \mathcal{D}_{\ell j}\left(\{\hat{p}_i\}\right) | q,r,s \rangle F_J^{(3)}(q,r,s) \right]_B \\
&+ \int dPS_3 \left\{ \left| \mathcal{M}_V\left(\{p_i\}\right) \right|^2 + \frac{1}{2} \left[\sum_{\ell,j} \mu^{2\varepsilon} \int d\xi_p \langle 1,2,3 | \mathcal{D}_{\ell j}\left(\{\hat{p}_i\}\right) | 1,2,3 \rangle \right]_A \right. \\
&\left. + \sum_{\text{flavours}} \frac{1}{4} \left[\sum_{\ell,j} \mu^{2\varepsilon} \int d\xi_p \langle 1,2,3 | \mathcal{D}_{\ell j}\left(\{\hat{p}_i\}\right) | 1,2,3 \rangle \right]_B \right\} F_J^{(3)}\left(\{p_i\}\right)
\end{aligned}
\tag{73}
$$

where ℓ,j sum over all possible pairs in the real emission phase space, $\hat{p}_i$ and p_i denote momenta belonging to Born- and real emission kinematics, and where $q,r,s \in \{1,2,3\}$ denote the mapped momenta in the real emission subtraction terms for the Born-type matrix elements. The factors $\frac{1}{2}$, $\frac{1}{4}$ in the integrated subtraction terms correspond to the process-dependent symmetry factors in the real emission contributions. In the symbolic notation above, $\int dPS_n$ contains all symmetry and flux factors for the respective phase space[10].

4.1 Tree level result

In the following, we follow the procedure of [63], *i.e.* we normalize our observables according to

$$
\frac{1}{\sigma_0}\, d\sigma_J^{\text{NLO}},
$$

---

[10]Note that $\int dPS_3$, $\int dPS_4$ are defined slightly differently in [65, 90].

where σ_0 denotes the total cross section for the process

$$e^+ e^- \longrightarrow q\,\bar{q} \tag{74}$$

given by [63]

$$\sigma_0 = \frac{4\,\pi\,\alpha^2}{3\,s}\,C_A\,q_f^2$$

for a quark with flavour f and charge q_f. We normalize the matrix elements according to

$$\sigma_J^{(n)} = \frac{1}{2\,s}\,S\int d\Gamma_n\,|\mathcal{M}_n|^2\,F_J^{(n)},$$

where $d\Gamma_n = \prod_i \left[\frac{d^4 p_i}{(2\pi)^4}\,\delta\left(p_i^2 - m_i^2\right)\right]\delta^{(4)}\left(\sum_{\rm in} p_{\rm in} - \sum_i p_i\right)$, s denotes the center-of-mass enegy and with

$$S = \begin{cases} 1/2! & \text{for} \quad \gamma^* \to q\,\bar{q}\,g\,g \\[2mm] 1/2! \times 1/2! & \text{for} \quad \gamma^* \to q\,\bar{q}\,q\,\bar{q} \end{cases}$$

being the symmetry factor of the respective process. We obtain the well-known relations between the matrix elements of the processes (74) and (71) [63]:

$$\frac{1}{4}|\mathcal{M}_3|^2 = \frac{8\,\pi\,\alpha_s}{s}\,C_F\,\frac{x_1^2 + x_2^2}{(1 - x_1)\,(1 - x_2)}\,\frac{1}{4}\langle|\mathcal{M}_2|^2\rangle$$

where $|\mathcal{M}_2|^2$ has been averaged over the emission angles, as well as

$$\sigma_{3,J} = \frac{\alpha_s}{2\,\pi}C_F\int dx_1 dx_2 \Pi_i\left[\Theta(1 - x_i)\Theta(x_i)\right]\Theta(x_1 + x_2 - 1)\frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}\sigma_0 F_J^{(3)}(x_1, x_2)$$

for jet observables. The gluon-helicity dependent squared matrix element for the leading order process (71) is given by [90]

$$\mathcal{T}^{\mu\nu}(p_1, p_2, p_3) = -\frac{1}{x_1^2 + x_2^2}\,|\mathcal{M}_3(p_1, p_2, p_3)|^2\,T^{\mu\nu}, \tag{75}$$

where

$$\begin{aligned} T^{\mu\nu} = {}& +2\frac{p_1^\mu p_2^\nu}{Q^2} + 2\frac{p_2^\mu p_1^\nu}{Q^2} - 2\frac{1 - x_1}{1 - x_2}\frac{p_1^\mu p_1^\nu}{Q^2} - 2\frac{1 - x_2}{1 - x_1}\frac{p_2^\mu p_2^\nu}{Q^2} \\ & -\frac{1 - x_1 - x_2 + x_2^2}{1 - x_2}\left[\frac{p_1^\mu p_3^\nu}{Q^2} + \frac{p_3^\mu p_1^\nu}{Q^2}\right] - \frac{1 - x_2 - x_1 + x_1^2}{1 - x_1}\left[\frac{p_2^\mu p_3^\nu}{Q^2} + \frac{p_3^\mu p_2^\nu}{Q^2}\right] \\ & +\left(1 + \frac{1}{2}\,x_1^2 + \frac{1}{2}\,x_2^2 - x_1 - x_2\right)g^{\mu\nu}. \end{aligned} \tag{76}$$

$\mathcal{T}_{\mu\nu}$ obeys $g^{\mu\nu}\mathcal{T}_{\mu\nu} = -|\mathcal{M}_3|^2$.

The virtual one-loop matrix element

The virtual contribution to the process (71) in the $\overline{\text{MS}}$ renormalization scheme is [63,65]

$$|\mathcal{M}_V(p_1, p_2, p_3)|^2 = |\mathcal{M}_3(p_1, p_2, p_3)|^2 \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)}$$

$$\times \left\{ -\frac{1}{\varepsilon^2} \left[(2C_F - C_A)\, y_{12}^{-\varepsilon} + C_A \left(y_{13}^{-\varepsilon} + y_{23}^{-\varepsilon} \right) \right] - \frac{1}{\varepsilon} \left(3\, C_F + \frac{11}{6} C_A - \frac{2}{3} n_f\, T_R \right) \right.$$

$$\left. + \frac{\pi^2}{2}\, (2\, C_F + C_A) - 8\, C_F \right\} + \frac{\alpha_s}{2\pi} \left[F(y_{12}, y_{13}, y_{23}) + \mathcal{O}(\varepsilon) \right], \tag{77}$$

where $F(y_{12}, y_{13}, y_{23})$ is defined in Eqn. (2.21) of [63].

4.2 Subtraction terms

In general, the subtraction terms are given by Eqn. (69) as

$$\mathcal{D}_{\ell j} \equiv \mathcal{D}_{f_\ell \hat{f}_\ell \hat{f}_j}(\hat{p}_\ell, \hat{p}_j) = \mathcal{D}_{f_\ell \hat{f}_\ell \hat{f}_j}^{\text{coll}}(\hat{p}_\ell, \hat{p}_j) + \delta_{\hat{f}_j, g} \sum_{k \neq (\ell, j)} \mathcal{D}^{\text{if}}(\hat{p}_\ell, \hat{p}_j, \hat{p}_k),$$

where the first part is the collinear subtraction term, and the second the sum over all interference terms. For the processes considered here, the interference terms only contribute to (A), while (B) only contains collinear divergences. In the following sections we will construct the subtraction terms for subprocesses (A) and (B), respectively.

4.2.1 Momentum mapping

For all subtraction terms, the m-parton phase space momenta are mapped as described in section 3.2.2; *i.e.* , for a splitting

$$p_\ell \rightarrow \hat{p}_\ell + \hat{p}_j,$$

we have

$$p_\ell = \frac{1}{\lambda} P_\ell - \frac{1 - \lambda + y}{2\, \lambda\, a_\ell}\, Q,$$

$$p_k = \Lambda(K, \hat{K})\, \hat{p}_k, \quad [k \neq (\ell, j)]$$

where Λ is defined according to Eqn. (35), and where

$$\hat{K} = Q - P_\ell, \; K = Q - p_\ell.$$

4.2.2 The subprocess: $\gamma^* \rightarrow q(\hat{p}_1)\, \bar{q}(\hat{p}_2)\, g(\hat{p}_3)\, g(\hat{p}_4)$

For this process, *ggg*, *qqg* collinear as well as interference terms need to be taken into account; these are given by Eqns. (60), (48), and (63) respectively. We have for the

collinear subtraction terms

$$\mathcal{D}_{qqg;\ell j}^{\text{coll}} = \frac{4\pi\alpha_s}{y\,(p_\ell\cdot Q)}\,C_F\left\{\frac{(\lambda-1+y)^2+4\,y}{4\,\lambda}\,F_{\text{eik}}+\frac{1}{2}\,z\,[1+y+\lambda]\right\},$$

$$\langle\nu|\mathcal{D}_{ggg;\ell j}^{\text{coll}}|\nu'\rangle\,\mathcal{T}^{\nu\nu'} = -2\pi\alpha_s\,C_A\,\frac{1}{(\hat{p}_\ell\cdot\hat{p}_j)^2}\,[2-\Delta(2-\Delta)]\,\frac{1}{x_1^2+x_2^2}|\mathcal{M}_3(p_1,p_2,p_3)|^2\,\times$$

$$\left\{\left[\frac{2}{Q^2}\left(2(k_\perp\cdot p_1)(k_\perp\cdot p_2)-\frac{1-x_1}{1-x_2}(k_\perp\cdot p_1)^2-\frac{1-x_2}{1-x_1}(k_\perp\cdot p_2)^2\right)\right.\right.$$

$$\left.\left.+k_\perp^2\left(1+\frac{x_1^2}{2}+\frac{x_2^2}{2}-x_1-x_2\right)\right]\right\},$$

with $k_\perp$ and Δ given by Eqns. (51) and (56). In total, we need to consider the following combinations:

$$\mathcal{D}_{qqg;13}^{\text{coll}},\ \mathcal{D}_{qqg;14}^{\text{coll}},\ \mathcal{D}_{qqg;23}^{\text{coll}},\ \mathcal{D}_{qqg;24}^{\text{coll}},\ \mathcal{D}_{ggg;34}^{\text{coll}},\ \mathcal{D}_{ggg;43}^{\text{coll}};$$

note especially that both combinations $(3,4),(4,3)$ need to be taken into account in the contributions from ggg splittings.

The interference terms are given by

$$\mathcal{D}_{\ell,j;k}^{\text{if}} = \Delta W_{\ell j}(k) = 4\pi\alpha_s\,C_{\ell k}\,\frac{2\,\hat{p}_\ell\cdot\hat{p}_k\,\hat{p}_\ell\cdot\hat{Q}}{\hat{p}_\ell\cdot\hat{p}_j\left(\hat{p}_j\cdot\hat{p}_k\,\hat{p}_\ell\cdot\hat{Q}+\hat{p}_\ell\cdot\hat{p}_j\,\hat{p}_k\cdot\hat{Q}\right)},\tag{78}$$

where we have the following contributions:

$$\mathcal{D}_{1,3;2}^{\text{if}},\ \mathcal{D}_{1,3;4}^{\text{if}},\ \mathcal{D}_{1,4;2}^{\text{if}},\ \mathcal{D}_{1,4;3}^{\text{if}},\ \mathcal{D}_{2,3;1}^{\text{if}},\ \mathcal{D}_{2,3;4}^{\text{if}},\ \mathcal{D}_{2,4;1}^{\text{if}},\ \mathcal{D}_{2,4;3}^{\text{if}},\ \mathcal{D}_{3,4;1}^{\text{if}},\ \mathcal{D}_{3,4;2}^{\text{if}},\ \mathcal{D}_{4,3;1}^{\text{if}},\ \mathcal{D}_{4,3;2}^{\text{if}}.$$

We want to emphasize again that there is *only one mapping* required for each (ℓ,j) pairing, *i.e.* we only have 5 independent mappings for the 12 subtraction terms listed above[11]. The subtracted cross section is then given by

$$\sigma^{R-A} = \int_4\left[d\sigma^R-d\sigma^A\right]$$

$$= \int dPS_4\left\{\,|\mathcal{M}_4(\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4)|^2\,F_4(\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4)\right.$$

$$\left.-\langle 1,2,3\,|\,\mathcal{D}_{13}+\mathcal{D}_{14}+\mathcal{D}_{23}+\mathcal{D}_{24}+\mathcal{D}_{34}+\mathcal{D}_{43}\,|\,1,2,3\,\rangle F_3(p_1,p_2,p_3)\,\right\}$$

$$\tag{79}$$

with the spin-averaged matrix element

$$\frac{1}{4}\,|\mathcal{M}_4(\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4)|^2 = \sigma_0\left(\frac{\alpha_s}{2\pi}\right)^2\,C_F\,4\,(4\,\pi)^5\,\times$$

$$\{(A+B+C)+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,3\leftrightarrow 4)\}\tag{80}$$

The quantities A,B,C are given in Appendix B of [63].

---

[11] The Catani Seymour prescription [65] requires 10 different mappings for this contribution.

4.2.3 The subprocess: $\gamma^* \rightarrow q(\hat{p}_1)\,\bar{q}(\hat{p}_2)\,q(\hat{p}_3)\,\bar{q}(\hat{p}_4)$

This process does not contain any soft/ interference singularities, and all subtraction terms are given by Eqn. (53). We need to keep track of the mother partons helicity in the subtraction term and obtain

$$\langle \nu | \mathcal{D}_{gqq;\ell j} | \nu' \rangle \, \mathcal{T}^{\nu\nu'} \;=\; 4\,\pi\,\alpha_s\,T_R\,\frac{1}{\hat{p}_\ell \cdot \hat{p}_j}\,|\mathcal{M}_3(p_1,p_2,p_3)|^2 \times$$

$$\left\{ 1 \,+\, \frac{2}{x_1^2 + x_2^2}\,\frac{1}{\hat{p}_\ell \cdot \hat{p}_j}\left[\frac{2}{Q^2}\left(2\,(k_\perp \cdot p_1)\,(k_\perp \cdot p_2) - \frac{1-x_1}{1-x_2}(k_\perp \cdot p_1)^2 - \frac{1-x_2}{1-x_1}(k_\perp \cdot p_2)^2 \right) \right.\right.$$

$$\left.\left. +\, k_\perp^2\left(1 + \frac{x_1^2}{2} + \frac{x_2^2}{2} - x_1 - x_2 \right) \right] \right\},$$

with $k_\perp$ and $k_\perp^2$ given by Eqns. (51) and (57) respectively.

We have to consider the following combinations

$$\mathcal{D}_{gqq;12},\ \mathcal{D}_{gqq;14},\ \mathcal{D}_{gqq;32},\ \mathcal{D}_{gqq;34} \tag{81}$$

and get

$$\sigma^{R-A} \;=\; \int_4 \left[d\sigma^R - d\sigma^A \right]$$

$$=\; \int dPS_4 \left\{ \, |\mathcal{M}_4(\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4)|^2 \; F_4(\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4) \right.$$

$$\left. -\langle 1,2,3 \,|\, \mathcal{D}_{13} + \mathcal{D}_{14} + \mathcal{D}_{32} + \mathcal{D}_{34} \,|\, 1,2,3 \,\rangle F_3(p_1,p_2,p_3) \, \right\} \tag{82}$$

with the spin averaged matrix element given by

$$\frac{1}{4}\,|\mathcal{M}_4(\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4)|^2 = \sigma_0 \left(\frac{\alpha_s}{2\,\pi}\right)^2 C_F\,4\,(4\,\pi)^5 \times$$

$$\{(D+E) + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4)\} \tag{83}$$

The quantities D and E are given in Appendix B of [63].

4.3 Integrated subtraction terms

4.3.1 Collinear integrals

The collinear integrals involve the $gq\bar{q}$, $qqg/\bar{q}\bar{q}g$ and ggg splittings:

$$I_{\text{coll}} \;=\; I_{\text{coll}}(gq\bar{q}) \,+\, I_{\text{coll}}(qqg) \,+\, I_{\text{coll}}(ggg). \tag{84}$$

After summing over all contributions, we obtain from Eqns. (54), (49) and (61)

$$\langle 1, 2, 3 \,|\, I_{\text{coll}}(gq\bar{q}) \,|\, 1, 2, 3 \rangle_B =$$
$$|\mathcal{M}_3(p_1, p_2, p_3)|^2 \frac{\alpha_s}{2\pi} T_R \left(\frac{4\pi\mu^2}{Q^2} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} n_f \times$$
$$\left[-\frac{2}{3\varepsilon} - \frac{16}{9} + \frac{2}{3} \left[(a_3 - 1) \ln(a_3 - 1) - (a_3 + 1) \ln a_3 \right] \right],$$

$$\langle 1, 2, 3 \,|\, I_{\text{coll}}(qqg) \,|\, 1, 2, 3 \rangle_A =$$
$$|\mathcal{M}_3(p_1, p_2, p_3)|^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \times$$
$$\frac{1}{2} \sum_{\ell=1,2} \left\{ -\frac{1}{\varepsilon} + 4 I_3(a_\ell) - \ln a_\ell + \frac{1}{2} \left[(9 - 7a_\ell)(a_\ell - 1) \ln(a_\ell - 1) + a_\ell(7a_\ell - 16) \log(a_\ell) \right. \right.$$
$$\left. \left. - 7 \log(y_{\max}(a_\ell)) - a_\ell(2y_{\max}(a_\ell) + 7) - 7y_{\max}(a_\ell) - 4 \right] \right\}$$

$$\langle 1, 2, 3 \,|\, I_{\text{coll}}(ggg) \,|\, 1, 2, 3 \rangle_A \rangle =$$
$$|\mathcal{M}_3(p_1, p_2, p_3)|^2 \frac{\alpha_s}{2\pi} C_A \left(\frac{4\pi\mu^2}{Q^2} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \times$$
$$\left\{ -\frac{1}{6\varepsilon} - \frac{4}{9} + \frac{1}{6} \left[(a_3 - 1) \ln(a_3 - 1) - (a_3 + 1) \ln a_3 \right] + I_{\text{fin}}(a_3) \right\}, \tag{85}$$

where all symmetry factors have already been taken into account. Here, I_{fin} and I_3 are given by Eqns. (62) and (50) respectively and need to be evaluated numerically.

4.3.2 Soft integrals

For a specific emitter/ spectator pair (p_ℓ, p_k), we obtain from Eqn. (64)

$$I_{\text{soft},\ell k} = \frac{4\pi\alpha_s}{2} C_{\ell k} \mu^{2\varepsilon} \int d\xi_p \, \Delta W_{\ell k} =$$
$$\frac{\alpha_s}{2\pi} C_{\ell k} \left(\frac{4\pi\mu^2}{Q^2} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \times \left\{ \frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon} \left[1 + \frac{1}{2} \ln \left[(\tilde{a}_0^{(\ell k)} + a_\ell) a_\ell \right] \right] - \frac{\pi^2}{6} + 3 \right.$$
$$+ \frac{1}{\pi} \left[\tilde{I}_{\text{fin}}^{(b)} \left(\frac{\tilde{a}_0^{(\ell k)}}{a_\ell} \right) + I_{\text{fin}}^{(d)}(\tilde{a}^{(\ell k)}, a_\ell) + I_{\text{fin}}^{(e)}(a_\ell) \right] + \frac{1}{4} \ln^2 \left[(\tilde{a}_0^{(\ell k)} + a_\ell) a_\ell \right]$$
$$\left. - \ln 2 \ln \left(\frac{\tilde{a}_0^{(\ell k)} + a_\ell}{a_\ell} \right) + 2 \ln a_\ell \right\}, \tag{86}$$

where

$$\tilde{I}_{\text{fin}}^{(b)}(b) \equiv I_{\text{fin}}^{(b)}(b) - \pi \left[\ln 2 \ln(1 + b) + \frac{1}{4} \ln^2(1 + b) \right].$$

The additional factor $\frac{1}{2}$ in the integrated subtraction terms stems from Eqn. (70) and accounts for the different symmetry factors and combinatorics of process (A). Soft

interference terms only appear for this process, where the following emitter/ spectator pairs need to be taken into account

$$2\left[(1,2)+(1,3)+(2,1)+(2,3)+(3,1)+(3,2)\right];\tag{87}$$

the factor 2 arises as e.g. $(\hat{p}_\ell, \hat{p}_j, \hat{p}_k) = [(\hat{p}_1\, \hat{p}_3,\, \hat{p}_2), (\hat{p}_1\, \hat{p}_4,\, \hat{p}_2)]$ are mapped to the same Born-type kinematics $(p_\ell, p_k) = (p_1, p_2)$, and similar relations hold for the other contributions.

4.3.3 Finite parts

Combining the one-loop matrix element Eqn. (77) with the integrated subtraction terms, all poles in ε cancel. The leftover finite parts are

$$< 1, 2, 3 \,|\, I_{\text{coll}} \,|\, 1, 2, 3 >_{\text{finite}} = \ |\mathcal{M}_3(p_1, p_2, p_3)|^2 \ \frac{\alpha_s}{2\,\pi} \ \times$$

$$\left\{ (4\,n_f\,T_R + C_A)\left[-\frac{4}{9} + \frac{1}{6}\left[(a_3 - 1)\ln(a_3 - 1) - (a_3 + 1)\ln a_3\right] \right] + C_A\,I_{\text{fin}}(a_3) \right.$$

$$+ \frac{C_F}{2} \sum_{\ell=1,2} \left[4\,I_3(a_\ell) - \ln a_\ell + \frac{1}{2}\left[(9 - 7a_\ell)(a_\ell - 1)\ln(a_\ell - 1) + a_\ell(7a_\ell - 16)\log(a_\ell) \right. \right.$$

$$\left. \left. \left. -7\log(y_{\text{max}}(a_\ell)) - a_\ell(2y_{\text{max}}(a_\ell) + 7) - 7y_{\text{max}}(a_\ell) - 4\right] \right] \right\} \tag{88}$$

and

$$< 1, 2, 3 \,|\, I_{\text{soft}} \,|\, 1, 2, 3 >_{\text{finite}} = \ |\mathcal{M}_3(p_1, p_2, p_3)|^2 \ \frac{\alpha_s}{2\,\pi} \ \times$$

$$\left\{ (2\,C_F + C_A)\left[-\frac{\pi^2}{3} + 6 \right] + 2 \sum_{\ell=1,2,3} C_\ell^2 \left[\frac{1}{\pi} I_{\text{fin}}^{(e)}(a_\ell) + 2\,\ln a_\ell \right] \right.$$

$$- 2\,(C_A - 2\,C_F) \left[\frac{1}{\pi} \tilde{I}_{\text{fin}}^{(b)}\left(\frac{\tilde{a}_0^{(1,2)}}{a_1} \right) - \ln 2\,\ln\left(\frac{\tilde{a}_0^{(1,2)}}{a_1} + 1 \right) \right]$$

$$+ 2\,C_A \sum_{\ell=1,2} \left[\frac{1}{\pi} \tilde{I}_{\text{fin}}^{(b)}\left(\frac{\tilde{a}_0^{(\ell,3)}}{a_\ell} \right) - \ln 2\,\ln\left(\frac{\tilde{a}_0^{(\ell,3)}}{a_\ell} + 1 \right) \right]$$

$$+ \sum_{(\ell,k)} C_{\ell k} \left[\frac{2}{\pi} I_{\text{fin}}^{(d)}\left(\tilde{a}^{(\ell,k)}, a_\ell \right) + \frac{1}{2}\,\ln^2\left[\left(\tilde{a}_0^{(\ell,k)} + a_\ell \right) a_\ell \right] \right] \right\}, \tag{89}$$

where the sum in the last line goes over all possible combinations as given in Eqn. (87) (the factor 2 is already accounted for), and where we made use of several symmetries[12].

---

[12]One useful relation is e.g.

$$\frac{\tilde{a}_0^{(m,n)}}{a_n} = \frac{\tilde{a}_0^{(n,m)}}{a_m}.$$

Further simplifications for the interference term finally render

$$
< 1, 2, 3 \, | \, I_{\text{soft}} \, | \, 1, 2, 3 >_{\text{finite}} = \; |\mathcal{M}_3(p_1, p_2, p_3)|^2 \, \frac{\alpha_s}{2\,\pi} \; \times
$$

$$
\left\{ (2\,C_F + C_A) \left[-\frac{\pi^2}{3} + 6 \right] + 2 \sum_{\ell=1,2,3} C_\ell^2 \left[\frac{1}{\pi} I_{\text{fin}}^{(e)}(a_\ell) + 2 \ln a_\ell \right] \right.
$$

$$
+ \frac{1}{2} C_A \left\{ 2 \left[\ln a_1 \, \ln a_2 - \ln a_3 \, \ln \left(\frac{a_1 \, a_2}{a_3} \right) \right] + \ln^2 y_{13} + \ln^2 y_{23} - \ln^2 y_{12} \right\}
$$

$$
+ C_F \left[\ln^2 \left(\frac{a_1}{a_2} \right) + \ln^2 y_{12} \right] + 2 \ln 2 \left[2\,C_F \, \ln (y_{12}\, a_1\, a_2) - C_A \, \ln \frac{y_{12}}{y_{13} y_{23} a_3^2} \right]
$$

$$
\left. + \frac{2}{\pi} \left[C_A \sum_{\ell=1,2} \tilde{I}_{\text{fin}}^{(b)} \left(\frac{1}{y_{\ell 3} a_\ell a_3} \right) - (C_A - 2 C_F) \, \tilde{I}_{\text{fin}}^{(b)} \left(\frac{1}{y_{12} a_1 a_2} \right) + \sum_{(\ell,k)} C_{\ell k} \, I_{\text{fin}}^{(d)} \left(\tilde{a}^{(\ell,k)}, a_\ell \right) \right] \right\}
$$

$$
(90)
$$

4.3.4   Final expressions

The finite parts of one-loop matrix element are given by

$$
|\mathcal{M}_V(p_1, p_2, p_3)|_{\text{finite}}^2 = |\mathcal{M}_3(p_1, p_2, p_3)|^2 \, \frac{\alpha_s}{2\pi} \; \times
$$

$$
\left\{ \frac{1}{2} \left[(C_A - 2\,C_F) \ln^2 y_{12} - C_A \left(\ln^2 y_{13} + \ln^2 y_{23} \right) \right] + \frac{\pi^2}{2} \left(2\,C_F + C_A \right) - 8\,C_F \right\}
$$

$$
+ \frac{\alpha_s}{2\pi} \left[F(y_{12}, y_{13}, y_{23}) + \mathcal{O}(\varepsilon) \right]
$$

$$
(91)
$$

If we combine the one-loop matrix element with the integrated subtraction terms, poles in ε exactly cancel, leading to finite results:

$$
\sigma^{V+A} = \int_3 \left[d\sigma^V + \int_1 d\sigma^A \right]
$$

$$
= \int dPS_3 \left\{ \langle 1, 2, 3 \, | I_{\text{soft}} | \, 1, 2, 3 \rangle + \langle 1, 2, 3 \, | I_{\text{coll}} | \, 1, 2, 3 \rangle + |\mathcal{M}_V(p_1, p_2, p_3)|^2 \right\} F_3(p_1, p_2, p_3)
$$

$$
= \int dPS_3 \left\{ \langle 1, 2, 3 \, | I_{\text{soft}} | \, 1, 2, 3 \rangle_{\text{fin}} + \langle 1, 2, 3 \, | I_{\text{coll}} | \, 1, 2, 3 \rangle_{\text{fin}} + |\mathcal{M}_V(p_1, p_2, p_3)|_{\text{fin}}^2 \right\} F_3(p_1, p_2, p_3)
$$

$$
(92)
$$

where

$$\langle 1,2,3 \,|I_{\text{soft}}|\, 1,2,3\rangle_{\text{fin}} + \langle 1,2,3 \,|I_{\text{coll}}|\, 1,2,3\rangle_{\text{fin}} + |\mathcal{M}_V(p_1,p_2,p_3)|^2_{\text{fin}}$$

$$= |\mathcal{M}_3(p_1,p_2,p_3)|^2 \, \frac{\alpha_s}{2\pi} \times$$

$$\left\{ (2\,C_F + C_A)\frac{\pi^2}{6} + 2\,C_F + \frac{50}{9}\,C_A - \frac{16}{9}\,n_f\,T_R + \frac{7}{2}\,C_F \ln(a_1\,a_2) \right.$$

$$+ \left(\frac{23}{6}\,C_A - \frac{2}{3}\,n_F\,T_R \right) \ln a_3 + (4\,n_f\,T_R + C_A) \left[\frac{1}{6}\,[(a_3 - 1)\ln(a_3 - 1) - a_3 \ln a_3] \right]$$

$$+ C_A\, I_{\text{fin}}(a_3) + \frac{C_F}{2} \sum_{\ell=1,2} \left[4\,I_3(a_\ell) + \frac{1}{2}\,[(9 - 7a_\ell)(a_\ell - 1)\ln(a_\ell - 1) + a_\ell(7a_\ell - 16)\log(a_\ell) \right.$$

$$\left. - 7\log(y_{\max}(a_\ell)) - a_\ell(2y_{\max}(a_\ell) + 7) - 7y_{\max}(a_\ell)] \right]$$

$$+ \frac{1}{2}C_A \left\{ 2 \left[\ln a_1 \ln a_2 - \ln a_3 \ln\left(\frac{a_1\,a_2}{a_3} \right) \right] \right\} + C_F \ln^2\left(\frac{a_1}{a_2} \right)$$

$$+ 2\ln 2 \left[2\,C_F \ln(y_{12}\,a_1\,a_2) - C_A \ln\frac{y_{12}}{y_{13}y_{23}a_3^2} \right] + \frac{2}{\pi} \sum_{\ell=1,2,3} C_\ell^2\, I_{\text{fin}}^{(e)}(a_\ell)$$

$$+ \frac{2}{\pi} \left[C_A \sum_{\ell=1,2} \tilde{I}_{\text{fin}}^{(b)}\left(\frac{1}{y_{\ell 3}a_\ell a_3} \right) - (C_A - 2\,C_F)\, \tilde{I}_{\text{fin}}^{(b)}\left(\frac{1}{y_{12}a_1 a_2} \right) + \sum_{(\ell,k)} C_{\ell k} I_{\text{fin}}^{(d)}\left(\tilde{a}^{(\ell,k)}, a_\ell \right) \right] \right\}$$

$$+ \frac{\alpha_s}{2\,\pi}\, F(y_{12}, y_{13}, y_{23}). \tag{93}$$

4.4   Results

We compared the implementation of our scheme with the results from [65,90] as well as our own implementation of the Catani Seymour scheme. The subtraction terms for the latter are well-known and will not be repeated here. In order to fulfill the jet-function requirements in Eqn. (10), we chose the variable [63],

$$C^{(n)} = 3 \left\{ 1 - \sum_{i,j\,=\,1,\, i<j}^{n} \frac{s_{ij}^2}{(2\,p_i \cdot Q)\,(2\,p_j \cdot Q)} \right\} \tag{94}$$

which is infrared finite as required[13]. We numerically compared all different color contributions $N_C\,C_F^2$, $N_c\,C_F\,n_f\,T_R$, $N_C^2\,C_F$ separately, as well as the combined contributions. We set $n_f = 5$ in our calculations. Figures 3 to 5 show that we agree with results obtained using the Catani Seymour subtraction scheme on percent-level, and are consistent with 0, within the errorbars, and thereby successfully validated our real emission subtraction terms as well as all interated counterterms proposed in this paper.

Results for integration as well as differential distributions have been obtained using routines from the Cuba library [93].

---

[13]A closer inspection of this variable shows that it contains singular regions which however are integrable [92]; we thank B. Webber for pointing this out.
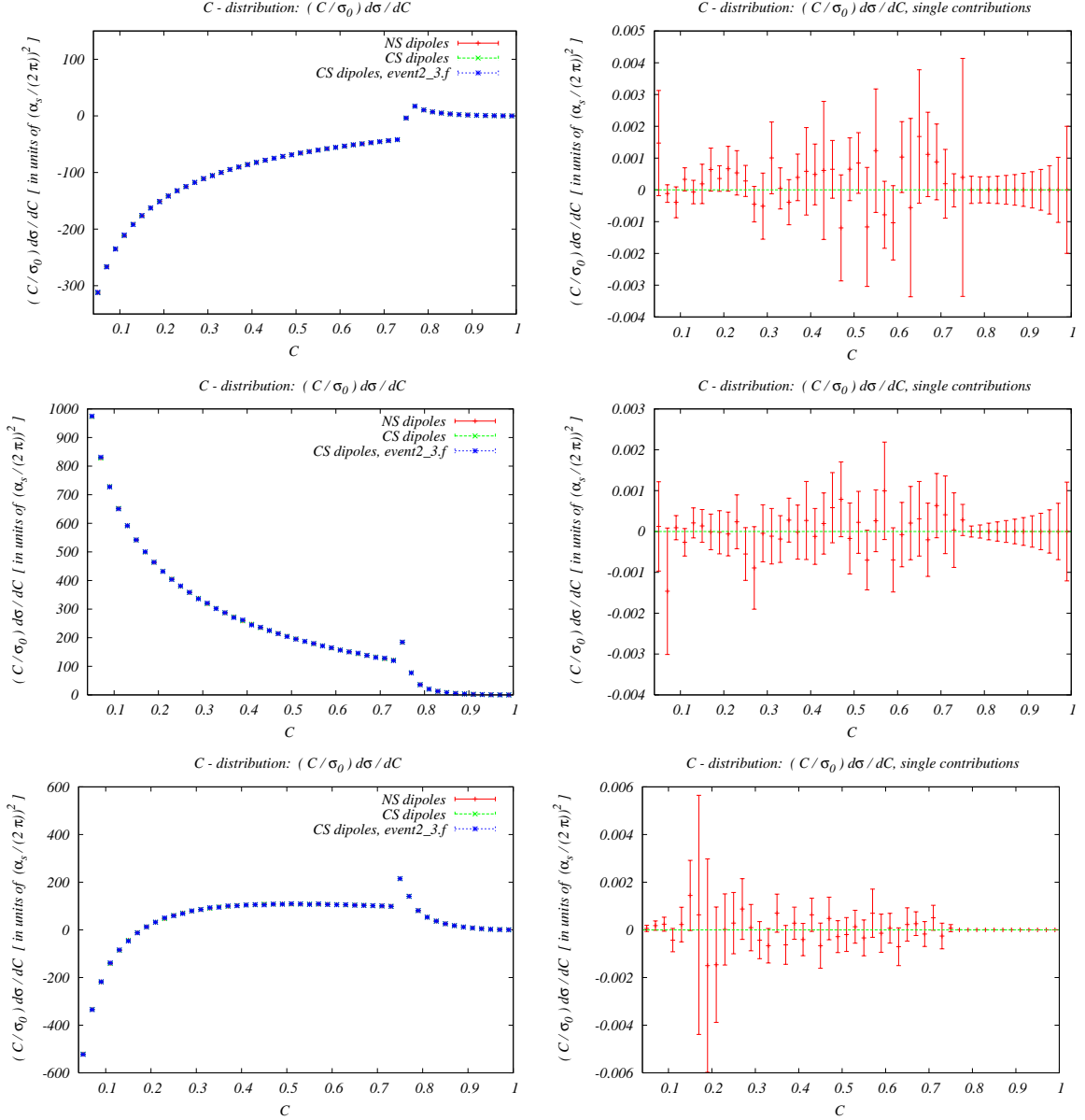
Figure 3: *From top to bottom:* Comparison of the contribution proportional to $N_C C_F n_f T_R$, $N_C^2 C_F$, $N_C C_F^2$ for our scheme (NS) and Catani Seymour dipoles (CS). Shown is the differential distribution $\frac{C}{\sigma_0} \frac{d\sigma^{NLO}}{dC}$ in units of $\left(\frac{\alpha_s}{2\pi}\right)^2$, with C defined by Eqn.(94). *Left:* Results for the implementation of our scheme and Catani Seymour subtraction terms from our private code as well as event2_3.f [91]. *Right:* relative difference between our implementation of the CS and NS subtraction terms. The results agree on the percent level and are consistent with zero within the integration errors. Large errors arize in regions where the absolute values of the differential distribution become small.
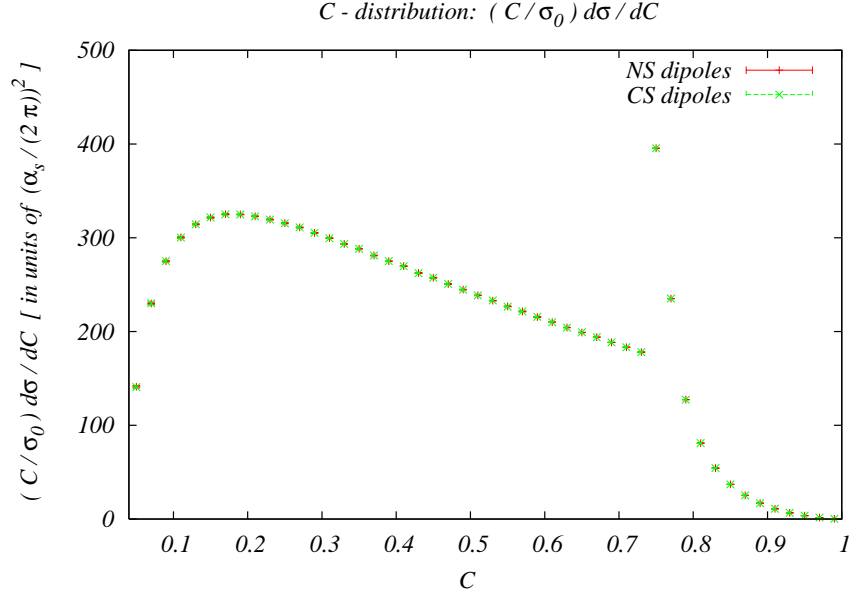
Figure 4: Total result for differential distribution $\frac{C}{\sigma_0} \frac{d\sigma^{NLO}}{dC}$ using both NS (red) and CS (green) dipoles. The standard literature result obtained using the CS scheme is completely reproduced with the NS dipoles.
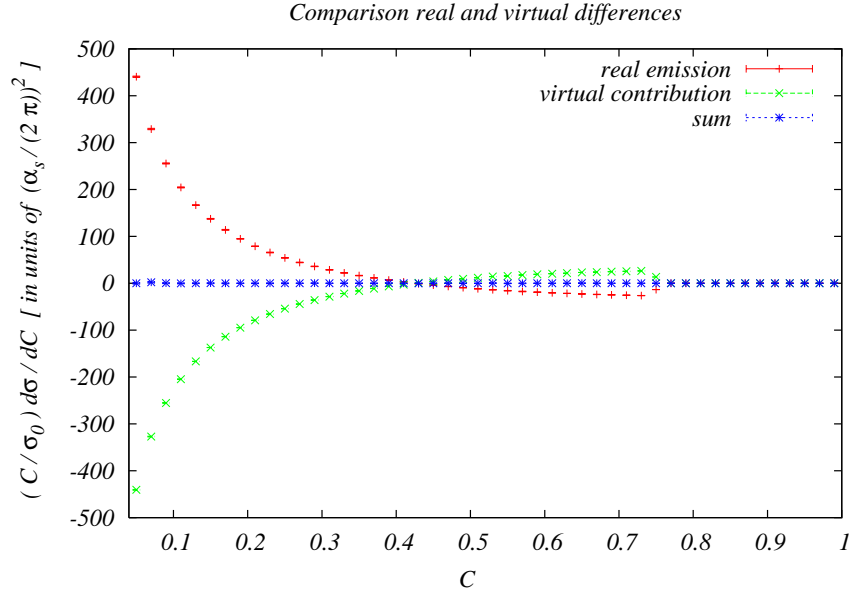

Figure 5: *Differences* Δ_{CS-NS} for real emission (red, upper) and virtual (green, lower) virtual contributions, showing that especially for low C values the contributions in the two schemes significantly differ. Adding up $\Delta^{\mathrm{real}} + \Delta^{\mathrm{virt}}$ gives 0 as expected.

31

5   Conclusion and Outlook

In this work, we have proposed an alternative next-to-leading order subtraction scheme, where the subtraction terms have been derived in the context of an improved parton shower. In this scheme, we employ a momentum mapping which reduces the number of reevaluations of the underlying Born matrix element with respect to standard schemes [65, 70] [14]. Furthermore, the use of subtraction terms based on the shower splitting functions promises to facilitate the matching of parton-level NLO-corrections with the improved parton shower. While final state subtraction terms for processes with maximally two partons in the final state as well as initial state subtractions were already presented in previous work [58–60], we here extend the scheme to processes with an arbitrary number of final state particles and provide formulae for the corresponding final state subtraction terms and their integrated counterparts. We validate our expressions by reproducing the literature results for the differential distribution of the C-parameter at NLO for three jet production at lepton colliders, where we find numerical agreement between results from the implementation of our scheme and two independent implementations of the Catani Seymour scheme on the (sub)percent level. Combining the results in this work with the discussion in [58–60] provides all formulae needed for a generic application of our scheme for massless emitters, and therefore concludes the discussion of the subtraction scheme in the massless case.

As argued in the introduction, subtraction schemes can generically differ in the non-singular structure of the dipole subtraction terms as well as the mapping between real emission and Born-type kinematics, which guarantees onshellness and energy momentum conservation in both phase spaces. The scheme proposed here uses the whole remaining event as a spectator for the mapping, thereby leading to a scaling behaviour of Born reevaluations $\sim N^2/2$, where N is the number of final state partons. However, this simplified mapping equally induces integrated subtraction terms with finite parts which exhibit a non-trivial dependence on the integration parameters of the unresolved one-parton phase space. In this work, we chose to evaluate these finite terms numerically, which leads to an increase of integration variables by two in the numerical implementation of the scheme. However, recently it was proposed [1] to approximate similar finite terms by polynomial functions in the context of an next-to-next-to-leading order subtraction scheme [94–97]. We therefore plan to make the finite remainders which appear in the integrated subtraction terms available either in form of approximating functions or a librarized grid interpolating between different input parameters for a_ℓ. Further plans for future work include the extension to the massive scheme as well as the matching with the improved parton shower[15].

---

[14]We want to note that the FKS subtraction scheme [64] exhibits a scaling behaviour similar to our scheme. However, the two prescriptions differ in the treatment of phase space setup; in addition, no parton shower proposal exists using FKS splitting functions. We thank R. Frederix for helpful discussions regarding this point.

[15]We note that work is underway [98] to implement the scheme presented here into the Helac NLO Event Generator framework [15]. We thank the authors for bringing this to our attention.

Acknowledgements

ℓ	f_ℓ	$\hat{f}_\ell$	$\hat{f}_j$		colour
ℓ	f_ℓ	$\hat{f}_\ell$	$\hat{f}_j$	$v_\ell \times \dfrac{1}{\sqrt{4\pi\alpha_s}}$	colour
F	q	q	g	$\varepsilon_\mu(\hat{p}_j,\hat{s}_j;\hat{Q})^* \dfrac{\overline{U}(\hat{p}_\ell,\hat{s}_\ell)\gamma^\mu[\not{\hat{p}}_\ell+\not{\hat{p}}_j]\eta_\ell U(p_\ell,s_\ell)}{2p_\ell\cdot n_\ell\,[(\hat{p}_\ell+\hat{p}_j)^2]}$	t^a
F	g	q	$\bar{q}$	$-\varepsilon^\mu(p_\ell,s_\ell;\hat{Q})D_{\mu\nu}(\hat{p}_\ell+\hat{p}_j,n_\ell)\dfrac{\overline{U}(\hat{p}_\ell,\hat{s}_\ell)\gamma^\nu V(\hat{p}_j,\hat{s}_j)}{(\hat{p}_\ell+\hat{p}_j)^2}$	t^a

Table 1: Splitting amplitudes $v_\ell(\{\hat{p},\hat{f}\}_j,\hat{s}_j,\hat{s}_\ell,s_\ell)$ involving a $q\bar{q}g$ splitting. We have removed a common factor $\sqrt{4\pi\alpha_s}$. The label ℓ denotes final-state indices $F=\{1,\dots,m\}$. The light-like vector n_ℓ is defined in Eqn. (97). Taken from [81].

A   Splitting amplitudes

In Table 1, we list the splitting amplitudes for final state $q\bar{q}g$ splittings as given in [81]. For triple gluon splittings, we have for the final state

$$
\begin{aligned}
v_\ell(\{\hat{p},&\hat{f}\}_{m+1},\hat{s}_j,\hat{s}_\ell,s_\ell)\\
&= \frac{\sqrt{4\pi\alpha_s}}{2\hat{p}_j\cdot\hat{p}_\ell}\,\varepsilon_\alpha(\hat{p}_j,\hat{s}_j;\hat{Q})^*\varepsilon_\beta(\hat{p}_\ell,\hat{s}_l;\hat{Q})^*\varepsilon^\nu(p_\ell,s_\ell;\hat{Q})\\
&\quad\times v^{\alpha\beta\gamma}(\hat{p}_j,\hat{p}_\ell,-\hat{p}_j-\hat{p}_\ell)\,D_{\gamma\nu}(\hat{p}_\ell+\hat{p}_j;n_\ell)\,.
\end{aligned}
\tag{95}
$$

We use standard notation where $U(p,s),\overline{U}(p,s),V(p,s),\overline{V}(p,s)$ denote spinors of the fermions with a four-momentum p and spin s, and $\varepsilon_\alpha(p,s;Q)$ are the gluon polarisation vectors. The ggg vertex has the form

$$
v^{\alpha\beta\gamma}(p_a,p_b,p_c) = g^{\alpha\beta}(p_a-p_b)^\gamma + g^{\beta\gamma}(p_b-p_c)^\alpha + g^{\gamma\alpha}(p_c-p_a)^\beta\,.
\tag{96}
$$

The transverse projection tensor $D_{\gamma\nu}(\hat{p}_\ell-\hat{p}_j;n_\ell)$ is defined according to Eqn. (22). The light-like vector n_ℓ is given by

$$
n_\ell = Q - \frac{Q^2}{Q\cdot p_\ell + \sqrt{(Q\cdot p_\ell)^2}}\,p_\ell\quad,\ell\in\{1,\dots,m\}\ .
\tag{97}
$$

B   Incoming hadrons

In case of processes with two initial-state hadrons A and B carrying momenta p_A and p_B, respectively, the calculation of the QCD cross sections must be convoluted with parton distribution functions $f_{i/I}(\eta_i,\mu_F^2)$ which depends on the factorization scale μ_F:

$$
\sigma(p_A,p_B) = \sum_{a,b}\int_0^1 d\eta_a\, f_{a/A}(\eta_a,\mu_F^2)\int_0^1 d\eta_b\, f_{b/B}(\eta_b,\mu_F^2)\left[\sigma_{ab}^{\mathrm{LO}}(p_a,p_b) + \sigma_{ab}^{\mathrm{NLO}}(p_a,p_b,\mu_F^2)\right]
$$

$$
\tag{98}
$$

where $p_a = \eta_a \, p_A$ and $p_b = \eta_b \, p_B$ are parton momenta, while η_a and η_b are the momentum fractions of the partons. In this case, additional collinear counterterms need to be added to the integrated subtraction terms[16], and parton level NLO contribution becomes

$$\sigma_{ab}^{\mathrm{NLO}}(p_a, p_b, \mu_F^2) = \int_{m+1} d\sigma_{ab}^R(p_a, p_b) + \int_m d\sigma_{ab}^V(p_a, p_b) + \int_m d\sigma_{ab}^C(p_a, p_b, \mu_F^2). \quad (99)$$

We then have

$$\sigma_{ab}^{\mathrm{NLO}}(p_a, p_b, \mu_F^2) = \int_{m+1} \left[d\sigma_{ab}^R(p_a, p_b) - d\sigma_{ab}^A(p_a, p_b) \right]$$
$$+ \int_m \left[\int d\sigma_{ab}^V(p_a, p_b) + \int_1 d\sigma_{ab}^A(p_a, p_b) + d\sigma_{ab}^C(p_a, p_b, \mu_F^2) \right]_{\varepsilon=0}, \quad (100)$$

with

$$\int_m \left[\int_1 d\sigma_{ab}^A(p_a, p_b) + d\sigma_{ab}^C(p_a, p_b, \mu_F^2) \right]$$
$$= \int_m d\sigma_{ab}^B(p_a, p_b) \otimes I(\varepsilon) + \int_0^1 dx \int_m d\sigma_{ab}^B(xp_a, p_b) \otimes \left[K^a(xp_a) + P(x, \mu_F^2) \right]$$
$$+ \int_0^1 dx \int_m d\sigma_{ab}^B(p_a, xp_b) \otimes \left[K^b(xp_b) + P(x, \mu_F^2) \right]. \quad (101)$$

This equation defines the insertion operators $I(\varepsilon)$, $K(x)$, $P(x; \mu_F^2)$ on an integrated cross section level. Eqn. (101) can be divided into two parts: the first part is the universal insertion operator $I(\varepsilon)$, which contains the complete singularity structure of the virtual contribution and has LO kinematics. The second part consists of the finite pieces that are left over after absorbing the initial-state collinear singularities into a redefinition of the parton distribution functions at NLO. It involves an additional one dimensional integration over the momentum fraction x of an incoming parton with the LO cross sections and the x-dependent structure functions.

In the $\overline{\mathrm{MS}}$ scheme, the collinear counterterms are given by

$$\int_m d\sigma_{ab}^C(p_a, p_b, \mu_F^2) = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \sum_c \int_0^1 dx \int_m d\sigma_{cb}^B(xp_a, p_b) \frac{1}{\varepsilon} \left(\frac{4\pi\mu^2}{\mu_F^2} \right)^\varepsilon P^{ac}(x)$$
$$+ \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \sum_c \int_0^1 dx \int_m d\sigma_{ac}^B(p_a, xp_b) \frac{1}{\varepsilon} \left(\frac{4\pi\mu^2}{\mu_F^2} \right)^\varepsilon P^{bc}(x). \quad (102)$$

Here the $P^{ab}(x)$ are the Altarelli-Parisi kernels in four dimensions [78], which are evolution kernels of the DGLAP equation [78, 100–102], and describe the behaviour of parton splittings by giving the probability of finding a parton of type b with momentum fraction x in a parton of type a in the collinear limit:

$$a(p) \longrightarrow b\left(x\,p + k_\perp + \mathcal{O}(k_\perp^2) \right) + c\left((1-z)\,p - k_\perp + \mathcal{O}(k_\perp^2) \right). \quad (103)$$

---

[16]C.f. e.g. [99].

At leading order, the splitting functions are given by

$$
\begin{aligned}
P^{qq}(x) &= C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right], \\
P^{gq}(x) &= T_R \left[x^2 + (1-x)^2 \right], \ T_R = \frac{1}{2}, \\
P^{qg}(x) &= C_F \left[\frac{1+(1-x)^2}{x} \right], \\
P^{gg}(x) &= 2\,C_A \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x\,(1-x) \right] + \delta(1-x)\frac{11\,C_A - 4\,n_f\,T_R}{6},
\end{aligned}
$$

$$(104)$$

where n_f is the number of quark flavours in the theory. The $+$ distribution is defined in the standard way

$$
\int_0^1 f(x)\,g_+(x)\,dx = \int_0^1 g(x)\,(f(x) - f(1))\,dx = \int_0^1 g(x)\,f(x)\,dx - f(1) \int_0^1 g(x)\,dx
$$

$$(105)$$

for the convolution with a test function $f(x)$.

C   Four-particle phase space

In this section, we derive the parametrization which was used for the real emission phase space in Section 4. We use the standard notation for an n-parton phase space in four dimensions:

$$
d\Gamma_n = \prod_i \left[\frac{d^4 p_i}{(2\pi)^4}\,\delta\left(p_i^2 - m_i^2\right) \right] \delta^{(4)}\left(p_{\text{in}} - \sum_i p_i \right).
$$

We build our parametrization from a successive chain with

$$
p_{\text{in}} \to p_{12} + p_{34}, \ p_{ij} \to p_i + p_j,
$$

where in the first step the onshell-condition for p_{ij} needs to be replaced by a distribution of $s_{ij} = (p_i + p_j)^2$.

Generic massive two parton phase space, center-of-mass system

For a generic massive two parton phase space, we use the following parametrization in the center-of-mass system

$$
d\Gamma_2 = \frac{1}{32\,\pi^2} \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{s} d\Omega_1\,\Theta\left(\sqrt{s} - (m_1 + m_2)\right),
$$

where

$$
p_1^0 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \ |\vec{p}_1| = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{2\sqrt{s}}, \ p_2 = p_{\text{in}} - p_1.
$$

The Θ function arizes from the conditions $\Theta(|\vec{p}_1|)\,\Theta(|\vec{p}_2|)$. λ is defined as

$$
\lambda(x_1, x_2, x_3) = \sum_i x_i^2 - 2 \sum_{i>j} x_i x_j.
$$

Generic massless two parton phase space, non-center-of-mass system

If we consider two partons in a non center-of-mass system, with

$$p_{\text{in}} = \begin{pmatrix} E_{\text{in}} \\ 0 \\ 0 \\ |\vec{p}_{\text{in}}| \end{pmatrix}$$

i.e. where the three vector of p_{in} determines the z-axis, we obtain for a two parton massless phase space

$$d\Gamma_2 = \frac{1}{16\,\pi^2\,|\vec{p}_{\text{in}}|}dp_1^0\,d\phi_1,$$

where we have

$$\cos\theta_1 = \frac{1}{|\vec{p}_{\text{in}}|}\left(E_{\text{in}} - \frac{m_{\text{in}}^2}{2\,p_1^0}\right) \tag{106}$$

and

$$\frac{m_{\text{in}}^2}{2\,(E_{\text{in}} + |\vec{p}_{\text{in}}|)} \leq p_1^0 \leq \frac{m_{\text{in}}^2}{2\,(E_{\text{in}} - |\vec{p}_{\text{in}}|)}$$

from the requirement that $|\cos\theta_1| \leq 1$. In this derivation,

$$p_{1,z} = p_1^0\,\cos\theta_1.$$

If the z-component of p_{in} goes into the negative z-direction, $\cos\theta \to -\cos\theta$ in Eqn. (106), and all other above relations still hold.

Generic four parton phase space with massless final states

We use the generic expression

$$d\Gamma_n\left(X \to \sum p_n\right) = d\Gamma_{X\to Y+Z}\,\frac{dm_X^2}{2\pi}\,\frac{dm_Y^2}{2\pi}\,d\Gamma_{X\to\sum p_o}\,d\Gamma_{Y\to\sum p_o'},$$

where $\sum p_o + \sum p_o' = \sum p_n$ is the sum over all n outgoing particles. Using the expressions above as well as

$$y_{ij} = \frac{s_{ij}}{s}, \quad x_i = 2\frac{p_i \cdot Q}{s},$$

we obtain for the four-parton phase space in the center-of-mass system of $p_{\text{in}} \equiv Q$:

$$d\Gamma_4 = \frac{s^2}{(4\,\pi)^6\,\sqrt{\lambda(1, y_{12}, y_{34})}}\,dy_{12}\,dy_{34}\,dx_1\,dx_3\,d\phi_3,$$

with the four-vectors

$$p_{12} = \begin{pmatrix} E_X \\ 0 \\ 0 \\ p_X \end{pmatrix}, \quad p_{34} = \begin{pmatrix} E_Y \\ 0 \\ 0 \\ -p_X \end{pmatrix},$$

$$p_1 = x_1\,\frac{\sqrt{s}}{2}\begin{pmatrix} 1 \\ \sin\theta_1 \\ 0 \\ \cos\theta_1 \end{pmatrix}, \quad p_3 = x_3\,\frac{\sqrt{s}}{2}\begin{pmatrix} 1 \\ \sin\theta_3\,\cos\phi_3 \\ \sin\theta_3\,\sin\phi_3 \\ \cos\theta_3 \end{pmatrix},$$

$$p_2 = p_{12} - p_1, \quad p_4 = p_{34} - p_3,$$

and where

$$E_X = \frac{\sqrt{s}}{2}\left(1 + y_{12} - y_{34}\right), \; p_X = \frac{\sqrt{s\,\lambda(1, y_{12}, y_{34})}}{2},$$

$$E_Y = \sqrt{s} - E_X = \frac{\sqrt{s}}{2}\left(1 + y_{34} - y_{12}\right),$$

$$\cos\theta_1 = \frac{1}{p_X}\left(E_X - \frac{m_X^2}{x_1\sqrt{s}}\right) = \frac{1}{\sqrt{\lambda(1, y_{12}, y_{34})}}\left(1 + y_{12} - y_{34} - \frac{2\,y_{12}}{x_1}\right),$$

$$\cos\theta_3 = -\frac{1}{p_X}\left(E_Y - \frac{m_Y^2}{x_3\sqrt{s}}\right) = -\frac{1}{\sqrt{\lambda(1, y_{12}, y_{34})}}\left(1 + y_{34} - y_{12} - \frac{2\,y_{34}}{x_3}\right).$$

The integration boundaries are given by

$$y_{34} \leq \left(1 - \sqrt{y_{12}}\right)^2,$$

$$x_1^{\min/\max} = \frac{m_X^2}{\sqrt{s}\,(E_X \pm p_X)} = \frac{2\,y_{12}}{1 - y_{12} + y_{34} \pm \sqrt{\lambda(1, y_{12}, y_{34})}},$$

$$x_3^{\min/\max} = \frac{m_Y^2}{\sqrt{s}\,(E_Y \pm p_X)} = \frac{2\,y_{34}}{1 - y_{34} + y_{12} \pm \sqrt{\lambda(1, y_{12}, y_{34})}}.$$

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